HAMILTONICITY OF VERTEX-TRANSITIVE GRAPHS

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Announcing two opening positions at UP Famnit / Pint in 2011/12

- Young Research position $\cong$ PhD Grant
- PostDoc / Visiting Professor position(s)

But why would one want to do a PhD in mathematics (among all other possibilities)

Crispin Nash-Williams

Reading University, Friday, October 7, 1977

Lovasz problem, 1969

Let us construct a finite, connected, undirected graph which is symmetric and has no simple path containing all vertices.

Cayley graph version

Folklore conjecture

Every connected Cayley graph has a Hamilton cycle.

Known results about hamiltonicity of VTGs

- VTG of order p, $2p$, $3p$, $4p$, $5p$, $6p$, $2p^2$, p^k (for $k \leq 4$), (Alspach, Chen, Du, Marušič, Parsons, Šparl, KK, etc.);
- VTG having groups with a cyclic commutator subgroup of order p^k (Durenberger, Gavlas, Keating, Marušič, Morris, Morris-Witte, etc.).

(p a prime)

How does one prove/disprove the existence of Hamilton cycles/paths in vertex-transitive graphs?

Sufficient knowledge about the structure of a graph is needed, where "sufficient" depends on the particular graphs in question.

For example, how does one prove the existence of Hamilton cycles in VTGs of prime order.

Description:

X := VTG order $2p$
Sylow p-subgroup P of $Aut(X)$ contains a semiregular automorphism π which gives rise to an algebraic description of

$$X := [R, S, T] \text{ where } R, S \subseteq \mathbb{Z}_p^* \text{ and } R \subseteq \mathbb{Z}_p.$$

Hamilton cycles in VTGs of order $2p$

Existence of a Hamilton cycle in VTGs of order $2p$, p a prime, proven in

B. Alspach, Hamiltonian cycles in vertex-transitive graphs of order $2p$, Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1979), pp. 131–139, *Congress. Numer.*, XXIII–XX, Utilitas Math., Winnipeg, Man., 1979.

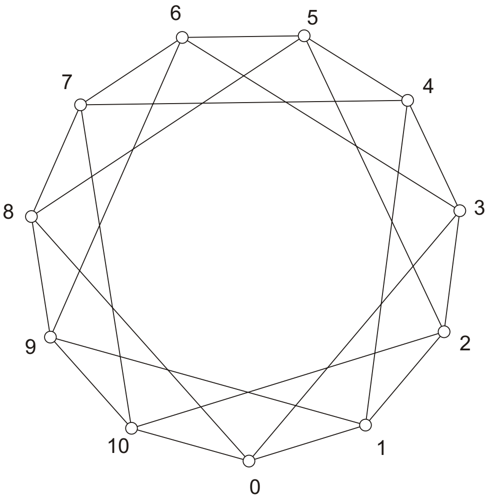
Hamilton cycles in VTGs of order $2p$

The proof contains two crucial steps:

- Step 1: a p-circulant of valency ≥ 4 is Hamilton-connected (for any two vertices u, v there exists a hamiltonian $(u - v)$-path).
- Step 2: every vertex-transitive generalized Petersen graph $GP(p, k)$ is hamiltonian (with the exception of the Petersen graph).

Step 1

$p = 11,\ X = \mathrm{Cay}(\mathbb{Z}_{11}, \{\pm 1, \pm 3\})$



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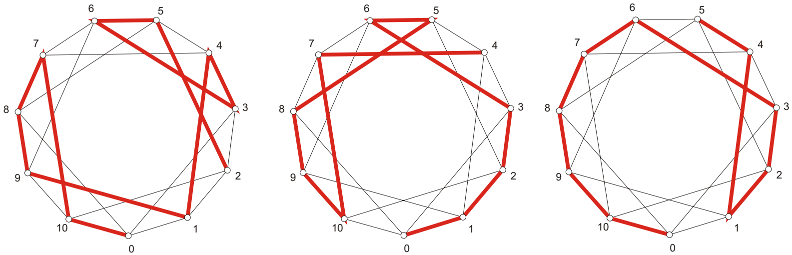


Step 1

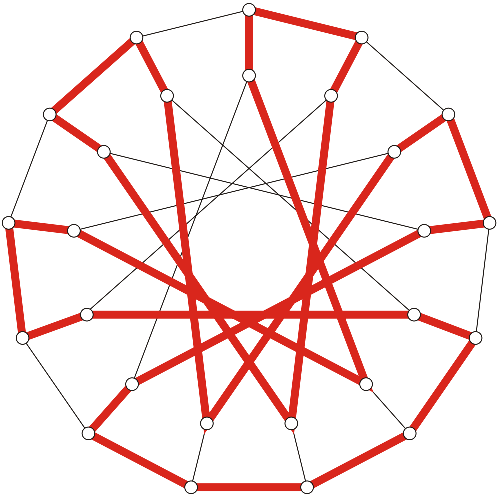
$X := [R, S, T]$ vertex-transitive graph of order $2p$.

Existence of a Hamilton cycle is clear when

- $|R| = |S| \geq 4$; or
- $|T| \geq 2$

This leaves us with the case where X is a generalized Petersen graph $GP(p, k)$, $S = \{\pm 1\}$, $R = \{\pm k\}$ and $T = \{0\}$ (because of vertex-transitivity we need additional assumption that $k^2 \equiv \pm 1 \pmod{p}$).

$GP(13, 5)$



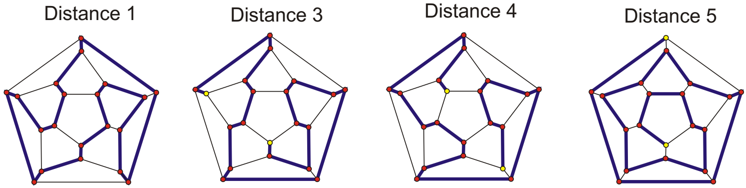
Generalization

Chen-Quimpo, 1981

A connected Cayley graph on an abelian group of valency ≥ 3 is Hamilton-connected or Hamilton-laceable.

The Dodecahedron

When non-abelian groups are involved hamiltonian connectedness is not necessary true.



Distance 2 ?

Generalized Petersen graphs

Generalized Petersen graphs $GP(n,k)$ with Hamilton cycles are completely classified (finished by Alspach):

$GP(n,2)$, $n \equiv 5 \pmod 6$ are the only exceptions.

Hamilton cycles in VTGs of order pq, p and q primes

Classification of VTG of order pq, $p > q$

From independent work of Praeger-Xu-Wang and DM-Scapellato

A vertex-transitive graph of order pq must be one of the following:

- a metacirculant,
- a certain family of graphs obtained as generalized $\mathbb{Z}_q$-covers of K_p, p a Fermat prime,
- a generalized orbital graph associated with certain primitive groups.

VTG of order pq

X VTG of order pq, $G \leq \mathrm{Aut} X$ transitive

- G-imprimitive blocks of size p;
- G-imprimitive blocks of size q;
- G is primitive.

VTG of order pq

From DM, Scapellato, Combinatorica 1994:

X VTG of order pq, $G \leq \mathrm{Aut}X$ transitive

- G-imprimitive blocks of size p ($\Rightarrow$ metacirculants);
- G-imprimitive blocks of size q, but no other transitive group with blocks of size p exists ($\Rightarrow$ special graph associated with Fermat primes);
- G and every other transitive subgroup of the automorphism group is primitive (the possible groups are listed in the table).

Primitive groups of degree pq without imprimitive subgroups and with non-isomorphic generalized orbital graphs

soc G	(p, q)	*action*	*comment*
$P\Omega^+(2d, k)$	$(k^d + 1, \frac{k^{d-1}-1}{k-1})$	singular 1-spaces	d a power of 2, p a Fermat prime
M_{22}	$(11, 7)$	see Atlas	
A_7	$(7, 5)$	triples	
$PSL(2, 61)$	$(61, 31)$	cosets of A_5	
$PSL(2, q^2)$	$(\frac{q^2+1}{2}, q)$	cosets of $PGL(2, q)$	$q \geq 5$
$PSL(2, p)$	$(p, \frac{p+1}{2})$	cosets of D_{p-1}	$p \equiv 1 \, (mod \, 4)$, $p \geq 13$

Metacirculants

Definition

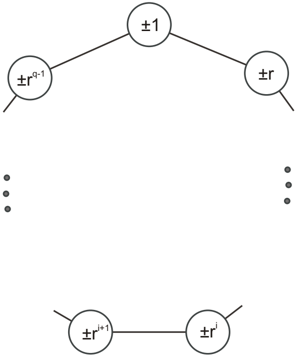
A graph X is an (m, n)-metacirculant if it admits a transitive group $G = \langle \rho, \sigma \rangle$, where ρ is a product of m disjoint copies of n-cycles and σ is a twister $\sigma^{-1}\rho\sigma = \rho^r$ (G is a semidirect product of $\langle \rho \rangle \rtimes \langle \sigma \rangle$).

Vertex-transitive graphs with p-blocks.

Step 1: May assume that the bipartite graph induced by two adjacent blocks (orbits of ρ) is a perfect matching. For otherwise a "spiral path" in the quotient lifts to a Hamilton cycle in the graph.

Step 2: Because of Chen-Quimpo result the degree of circulants induced on the blocks is 2, and the graph is a "twisted product" of C_p with a q-circulant, and therefore contains a twisted product of C_p with C_q.

"Twisted product" of C_p with C_q



A Hamilton cycle is obtained by glueing together Hamilton paths inside the orbits.

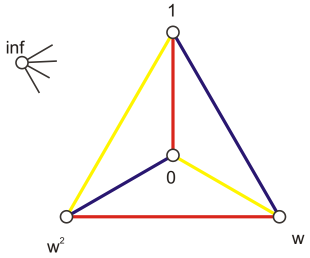
Every such graph X must be of the following form:

- $p = 2^{2^s} + 1$ is a Femat prime;
- q divides $p - 2$;
- X must contain a subgraph isomorphic to a $\mathbb{Z}_q$-cover of $Y \cong K_p$;
- $\rho \in S_q$ is the q-cycle $(0\,1\,\ldots\,q-1)$;
- $\tau \in S_q$ is the involution interchanging i with $-i$ $(i \in \mathbb{Z}_q)$;
- $\tau_i = \rho^i \tau \rho^{-i}$ $(i \in \mathbb{Z}_q)$;
- ω is a generator of the cyclic group $GF(2^{2^s})^*$.

$V(Y) = PG(1, 2^s) = GF(2^{2^s}) \cup \{\infty\}$.
The voltage function ζ satisfies:

$$\zeta(u,v) = \begin{cases} id; & u = \infty \text{ or } v = \infty \\ \tau_i; & u + v = \omega^i \end{cases}$$

The smallest example is the line graph of the Petersen graph with $q = 3$, $p = 5$



Red: τ_0
Yellow: τ_1
Blue: τ_2

Hamiltonicity of VTG of order pq, $p > q$

Based on the classification of vertex-transitive graphs of order pq, done independently Praeger, Xu, Wang and by DM, Scapellato (early 90s).

Almost Theorem (Du, Kutnar, DM)

A connected vertex-transitive graph of order pq other then the Petersen graph contains a Hamilton cycle.

- $\exists$ transitive $G \leq \mathrm{Aut} X$ with p-blocks
  $\Leftrightarrow X =$metacirculant (Alspach, Parsons, DM, early 80s)

- $\exists$ transitive $G \leq \mathrm{Aut} X$ with q-blocks (and no subgroup with p-blocks)
  $\Leftrightarrow X =$ Fermat graph ($p = 2^{2^s} + 1$, q divides $2^{2^s} - 1$, etc.)
  (DM, '92)

- all transitive subgroups of $\mathrm{Aut} X$ are primitive

Primitive groups of degree pq

Primitive groups of degree pq without imprimitive subgroups and with non-isomorphic generalized orbital graphs

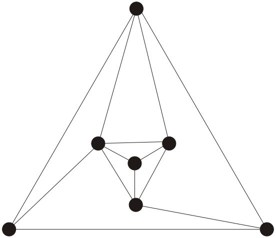
$soc\ G$	(p,q)	$action$	$comment$
$P\Omega^+(2d,k)$	$(k^d+1, \frac{k^{d-1}-1}{k-1})$	singular 1-spaces	d a power of 2 p a Fermat prime
M_{22}	$(11,7)$		
A_7	$(7,5)$	triples	
$PSL(2,61)$	$(61,31)$	cosets of A_5	
$PSL(2,q^2)$	$(\frac{q^2+1}{2}, q)$	cosets of $PGL(2,q)$	$q \geq 5$
$PSL(2,p)$	$(p, \frac{p+1}{2})$	cosets of D_{p-1}	$p \equiv 1\,(mod\,4)$ $p \geq 13$

Chvátal ideals

Chvátal, '72

Let X be a graph and let $S_i = \{x \in V(X) \mid deg(x) \leq i\}$. Then X has a HC if for each $i < n/2$ either $|S_i| \leq i - 1$ or $|S_{n-i-1}| \leq n - i - 1$.

Example



$$S_1 = \emptyset,\ S_2 = \emptyset \text{ and } |S_3| = 3$$

$$|S_3| \leq 2 \text{ or } |S_{7-3-1}| = |S_3| \leq 7-3-1 = 3$$

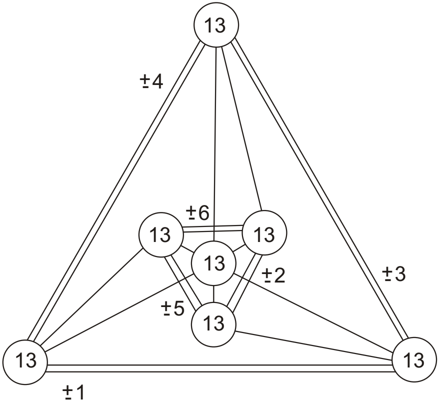
$\Rightarrow$ By Chvátal the graph has a Hamilton cycle.

Strategy

The strategy consists in:

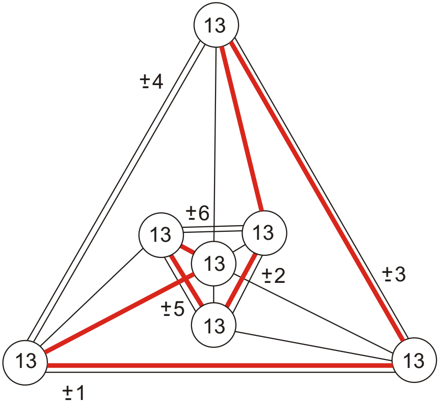
- first, quotiening the graph with respect to a semiregular automorphism of order p,
- second, using Chvátal theorem, finding a Hamilton cycle in the corresponding quotient graph of order q,
- third, lifting this cycle to a Hamilton cycle in the original graph.

$p = 13$ and $q = 7$



Example - A graph arising from the action of $PSL(2,13)$ on the cosets of D_{12}

$p = 13$ and $q = 7$



Open case

This approach works in all cases with the exception of the case when $G = PSL(2, p)$, $p \equiv 1 \pmod 8$ and the corresponding graph of order $p(p+1)/2$ has valency $(p-1)/4$.

Work in progress linking the existence of Hamilton cycles with certain number-theoretic conditions.

1. SMALL RANK GROUPS

soc G	(p, q)	*action*	*comment*
$P\Omega^+(2d, k)$	$(k^d + 1, \frac{k^{d-1}-1}{k-1})$	singular 1-spaces	d a power of 2 p a Fermat prime
M_{22}	$(11, 7)$	see Atlas	degrees: 60, 16
A_7	$(7, 5)$	triples	degrees: 4,12,18

• Every 2-connected regular graph of order n and valency at least $n/3$ is hamiltonian. [Jackson, 1978]

• Using Gap and Magma.

□

2. "SMALL" GRAPH

soc G	(p, q)	*action*	*comment*
$PSL(2, 61)$	$(61, 31)$	cosets of A_5	

- Using Gap and Magma (Conder, '03). □

3. ACTION OF $PSL(2, q^2)$

soc G	(p, q)	*action*	*comment*
$PSL(2, q^2)$	$(\frac{q^2+1}{2}, q)$	cosets of $PGL(2, q)$	$q \geq 5$

□

4. ACTION OF $PSL(2,p)$

$soc\ G$	(p,q)	$action$	$comment$
$PSL(2,p)$	$(p, \frac{p+1}{2})$	cosets of D_{p-1}	$p \equiv 1\,(mod\,4)$ $p \geq 13$

- For $q = \frac{p+1}{2}$ to be a prime we must have $p \equiv 1\,(mod\,4)$.
- Let $G = PSL(2,p)$, $H = D_{p-1}$, $F = GF(p)$ and $F^* = F \setminus \{0\}$.
- For simplicity reasons we refer to the elements of G as matrices.
- Let H consist of all the matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -x \\ x^{-1} & 0 \end{bmatrix} \ (x \in F^*).$$

- For a typical element

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{of } G \text{ we let} \quad \xi(g) = ad \quad \text{and} \quad \eta(g) = a^{-1}b.$$

- Let $\chi(g) = (\xi(g), \eta(g))$ be the character of g.
- Let $\sim$ be the equivalence relation on $F \times F^*$ defined by

$$(\xi, \eta) \sim (1 - \xi, \frac{\xi\eta}{\xi - 1}) \quad \text{for} \quad \xi \neq 0, 1.$$

- There is then a natural identification of the set of cosets $\mathcal{H}$ and $(F \times F^*)/_\sim \cup \{\infty\}$ where ∞ corresponds to H and (ξ, η) corresponds to the coset Hg satisfying $\chi(g) = (\xi, \eta)$.

4. ACTION OF $PSL(2, p)$

- Let S^* denote the set of all non-zero squares in F and let $N^* = F^* \setminus S^*$.
- For each $\xi \in S^*$ define the following subsets of $\mathcal{H}$:

$$\mathcal{S}_\xi^+ = \{(\xi, \eta) : \eta \in S^*\},$$

$$\mathcal{S}_\xi^- = \{(\xi, \eta) : \eta \in N^*\}$$

$$\mathcal{S}_\xi = \mathcal{S}_\xi^+ \cup \mathcal{S}_\xi^- .$$

- For $\xi \neq 0, 1$, the sets $\{\mathcal{S}_\xi^+, \mathcal{S}_\xi^-\}$ and $\{\mathcal{S}_{1-\xi}^+, \mathcal{S}_{1-\xi}^-\}$ coincide.

The following result determines the suborbits of the action of G on $\mathcal{H}$.

Theorem [Marušič and Scapellato, '92]
The action of G on $\mathcal{H}$ has

(i) $\frac{p+7}{4}$ suborbits of length $p-1$, all of them self-paired. These are $\mathcal{S}_0^+ \cup \mathcal{S}_1^+$, $\mathcal{S}_0^- \cup \mathcal{S}_1^-$ and $\mathcal{S}_\xi$ for all those ξ which satisfy $\xi^{-1} - 1 \in N^*$.

(ii) $\frac{p-5}{2}$ suborbits of length $\frac{p-1}{2}$, namely $\mathcal{S}_\xi^+$ and $\mathcal{S}_\xi^-$ where $\xi^{-1} - 1 \in S^*$. Among them the self-paired suborbits correspond to all those ξ for which both ξ and $\xi - 1$ belong to N^* and so their number is $\frac{p-9}{4}$ if $p \equiv 1\,(mod\,8)$ and $\frac{p-5}{4}$ if $p \equiv 5\,(mod\,8)$.

(iii) 2 suborbits of length $\frac{p-1}{4}$, namely $\mathcal{S}_{\frac{1}{2}}^+$ and $\mathcal{S}_{\frac{1}{2}}^-$ which are self-paired if and only if $p \equiv 1\,(mod\,8)$.

Each graph arising from the action of G on $\mathcal{H}$ is a generalized orbital graph $X = X(G, H, W)$ where W is a self-paired union of suborbits of G.

The generalized orbital graph $X = X(G,H,W)$, where W is a self-paired union of suborbits of G has vertex set

$$V(X) = \{Hg \mid g \in G\}$$

and edge set

$$E(X) = \{\{Hg, Hwg\} \mid g \in G, w \in W\}$$

The description of these graphs is best done via a factorization modulo the Sylow p-subgroup generated by the matrix

$$g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Example:
The smallest admissible pair of primes $p = 13$ and $q = (p+1)/2 = 7$ give rise to the following suborbits:

- $\mathcal{S}_0^+ \cup \mathcal{S}_1^+$, $\mathcal{S}_0^- \cup \mathcal{S}_1^-$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_5$ of size 12 – all of them self-paired;
- $\mathcal{S}_4^+$, $\mathcal{S}_4^-$ of size 6 – all of them self-paired;
- $\mathcal{S}_6^+$, $\mathcal{S}_6^-$ of size 6, – which are not self-paired;
- $\mathcal{S}_7^+$, $\mathcal{S}_7^-$ of size 3 which are not self-paired.

Each of the corresponding generalized orbital graphs is a union of the graphs $X(G, H, W)$ with W a self-paired union of the above suborbits of G.

Part (i) and (ii) of the theorem are solved using the following result due to Chvatal.

Theorem [Chvatal, '72]
Let X be a graph and let $S_i = \{x \in V(X) \mid deg(x) \leq i\}$. Then X has a Hamilton cycle if for each $i < n/2$ one of the following is true.

(i) either $|S_i| \leq i - 1$;

(ii) or $|S_{n-i-1}| \leq n - i - 1$.

Part (iii):

2 suborbits of length $\frac{p-1}{4}$, namely $\mathcal{S}^{+}_{\frac{1}{2}}$ and $\mathcal{S}^{-}_{\frac{1}{2}}$ which are self-paired if and only if $p \equiv 1 \,(mod\, 8)$.

$$\vdots$$