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Group actions and quotient graphs

Cheryl E Praeger

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Contribution to course by John Bamberg, Michael Giudici and Cheryl E. Praeger

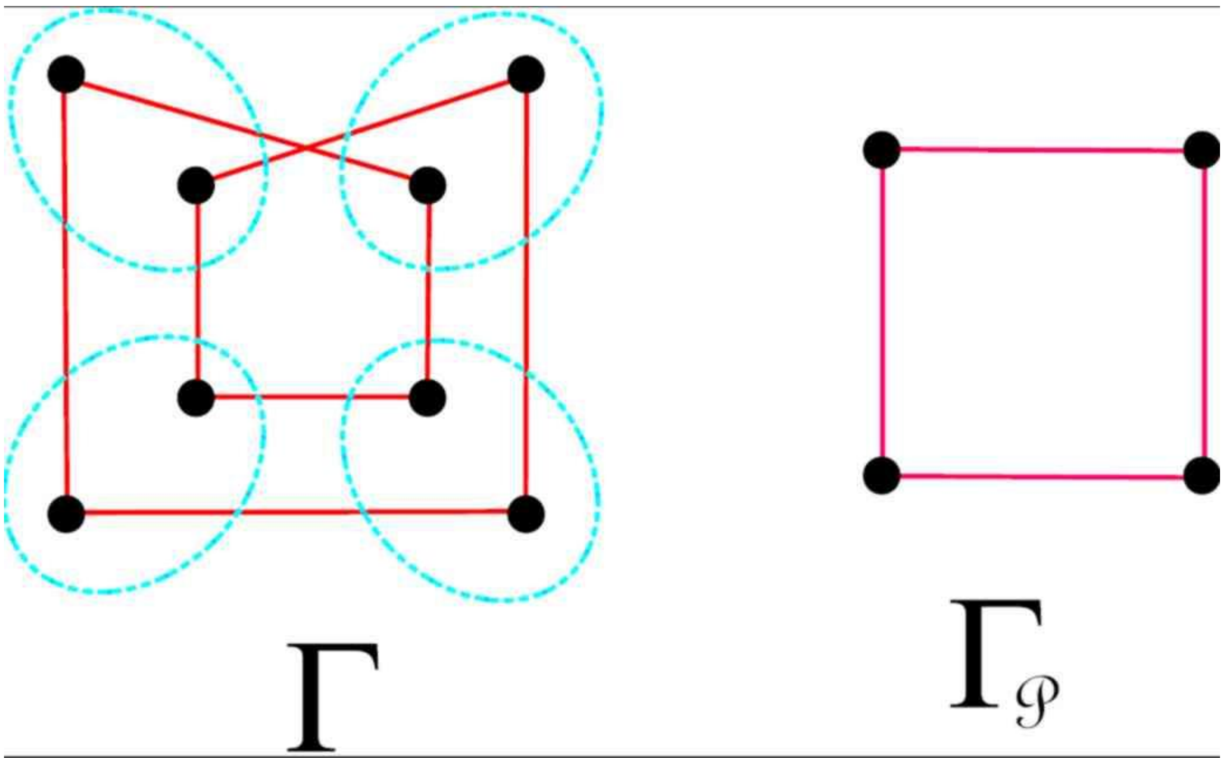
Graph Quotients: why?

- searching for graphs which are building blocks in graph families in analogy with
  - composition factors for finite groups
  - finite (quasi)primitive permutation groups

- want framework to study families of graphs
  - quotients should preserve desirable graph properties

Graph quotients: what are they?

- Take any vertex partition P of graph Γ
- Quotient modulo P has vertex set P, edges B, B' where some x in B and x' in B' forms an edge of Γ
- Note: connectivity preserved



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Imprimitive graph quotients

Suppose graph Γ admits automorphism group G

Choose partition P to be G-invariant $(B^g \in P \,\forall\, B \in P, g \in G)$

Quotient graph Γ_P: has $G^P \leq \mathrm{Aut}(\Gamma_P)$

In this case call Γ_P an imprimitive quotient of Γ

G transitive on $V\Gamma \Rightarrow G^P$ transitive on $V\Gamma_P$

G transitive on $E\Gamma \Rightarrow G^P$ transitive on $E\Gamma_P$

G transitive on arcs of $\Gamma \Rightarrow G^P$ transitive on arcs of Γ_P

Issue 1: degenerate quotients

$P = \{V\Gamma\}$ always G-invariant so K_1 always an imprimitive quotient

For Γ bipartite with bipartition $P = \{\Delta_1, \Delta_2\}$: then P is G=invariant and $\Gamma_P \cong K_2$ is an imprimitive quotient of Γ

Some issues:

- Want useful links between Γ and quotient
- links too weak for quotients K_1, K_2 - these considered degenerate quotients
- Want quotients a bit bigger than K_1, K_2 but 'not too much bigger'

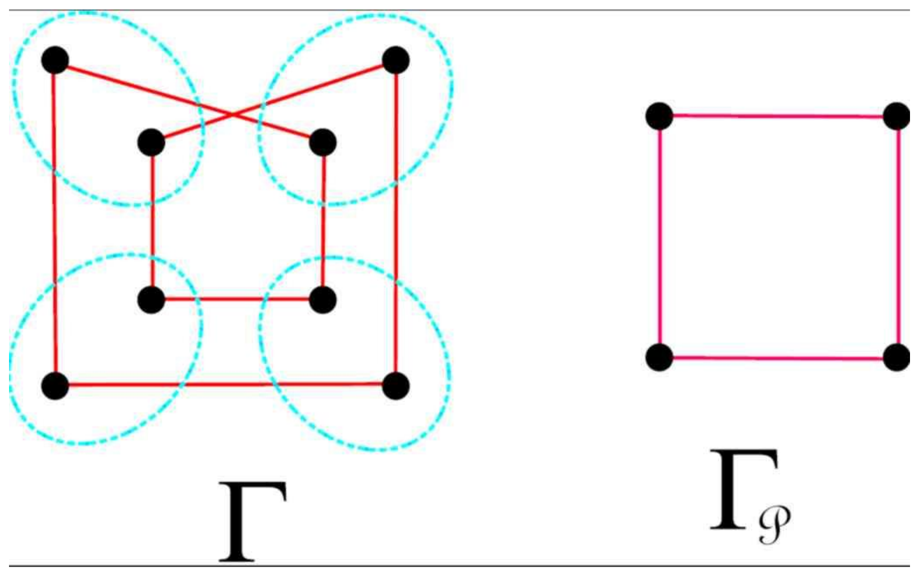
"Basic" imprimitive quotients

- For connected graphs and G vertex and edge transitive
- Definition: Want "basic" imprimitive quotients to be
  - bigger than K1 and K2 but
  - their only proper imprimitive quotients should be K1, K2
- If Γ is not bipartite then for basic imprimitive quotients induced G-action is vertex-primitive
- If Γ is bipartite then for basic imprimitive quotients induced G-action is vertex-biprimitive
- Exists block system with two blocks and induced action on blocks in primitive

Other issues:

- diameter may decrease (Ex: C_{2n} to quotient C_n)
- valency may increase out of control

Example: $F = \{2\text{–arc transitive graphs}\}$ (Michael will discuss). Many examples of imprimitive quotients where adjacent blocks have just 1 edge between them and quotients far from 2-arc transitive



G-normal quotients Γ for G in Aut(Γ)

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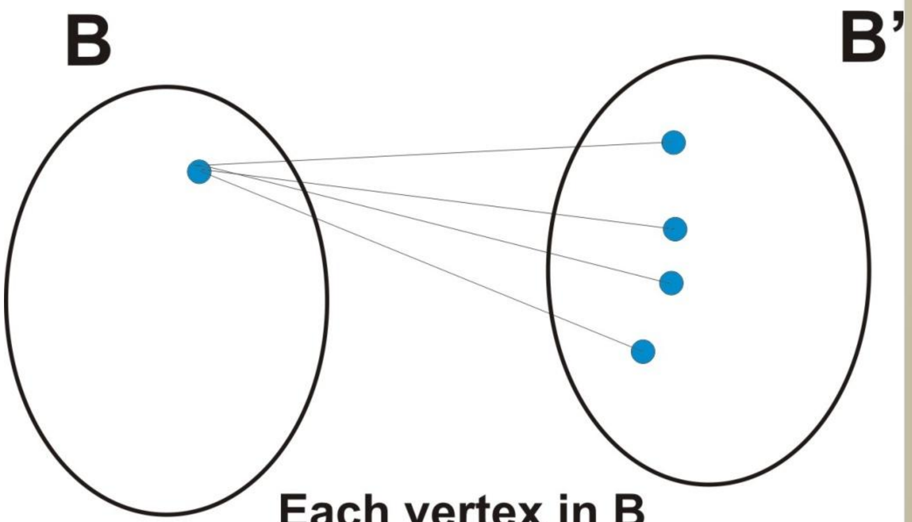
- N normal in G means
  - P = { N-orbits in V Γ } is G-invariant
  - Quotient of Γ modulo P called a G-normal quotient
  - Special kind of imprimitive quotient
- Properties of G-normal quotients when G edge-transitive

  ○ Γ multicover of normal quotient

  ○ valency controlled
  ○ properties like 2-arc transitivity preserved



Each vertex in B joined to same number of vertices in B'

Some principles for quotient analysis I

- Choice of type of quotient depends on
  - Property needing to be preserved
  - Desired links between graphs and their quotients
- Imprimitive quotients preserve connectivity, vertex-transitivity, edge-transitivity, arc-transitivity
- Normal quotients preserve more local properties, control valency - multicovers
- Quotients of half-transitive graphs may be arc-transitive instead of “half-transitive”

Some principles for quotient analysis II

- Define graph family F to be studied carefully
- Decide on quotient type
- Redefine F if necessary to ensure closure on forming quotients

Example 1: F-arc ={ connected arc-transitive graphs }

Quotient type: imprimitive quotients

Make sure F-arc contains K1 and K2

Degenerate and basic graphs in F

- Decide what graphs will be treated as degenerate in F – call the set Degen(F) – make sure they are all in F
- Define graph Γ in F to be basic if
  - Γ not in Degen(F) and
  - All proper quotients of Γ are in Degen(F)
- Example: F = F-arc Quotient type: imprimitive

What should be Degen(F)?

F = F-arc Quotient type: imprimitive

What should be Degen(F)?

First try: Degen(F) = { K1 }

Basics in F are: Γ where $|V\Gamma|>1$ and Aut(Γ) is arc-transitive and vertex-primitive. Is this OK?

Only basic bipartite graph in F is K2

F = F-arc Quotient type: imprimitive

What should be Degen(F)?

Second try: Degen(F) = { K1, K2 }

Basics in F now: vertex-primitive and vertex-biprimitive*

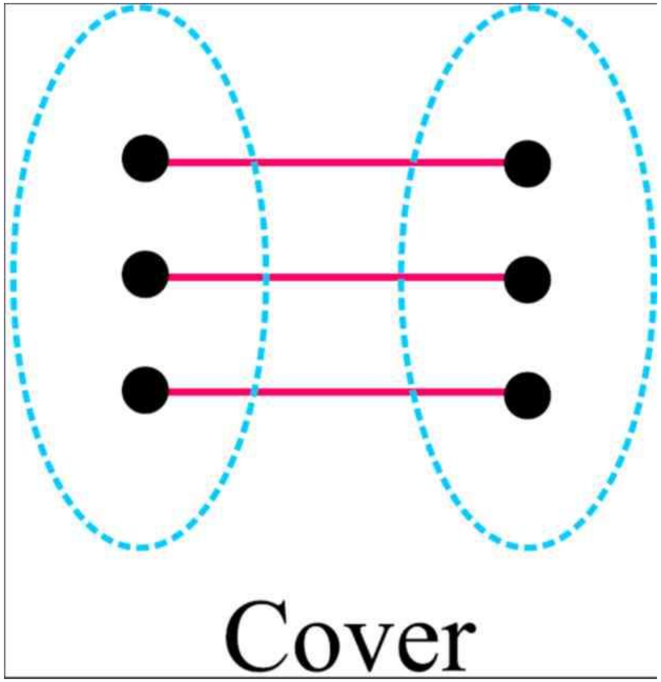
Often: this is OK – but do not have good control over valency, or "between block bipartite subgraphs"

* Only nontrivial invariant partition is the bipartition. Action on each block (bipart) is primitive.

F = { connected locally primitive }

- This means vertex stabiliser induces primitive action on neighbours
- Quotient type: imprimitive – no good – F not closed
  [imprimitive quotients usually not locally primitive]
- Quotient type: normal - good choice!
  - F closed
  - Γ a cover of its normal quotients*
  - Degen(F) = { K1, K2 }
  - Basics are: vertex-quasiprimitive
  - And vertex-biquasiprimitive

* Except the degenerate quotient K2



What is so special about vertex (quasi)primitive or bi(quasi)primitive?

- Answer from group theory: powerful methods from permutation groups
  - Allows use of finite simple group classification
  - Allows application of representation theory

Especially: O'Nan - Scott style theorems for these classes of (finite) permutation groups

Primitive case quote from web site of Scott’s papers

“....Often quoted O’Nan-Scott theorem, which described the types of maximal subgroups possible in the alternating groups. Actually, there were two theorems, and another gave an early version of the Aschbacher-Scott work. There was no collaboration between O’Nan and Scott, but both arrived at the 1979 Santa Cruz conference with similar results, and agreed to share credit. O’Nan never published separately. "Same damn theorem" was Michael O’Nan's comment after seeing the manuscript Scott had brought with him. In spite of this vote of confidence, that version turned out to have some inaccuracies ...”

Michael O'Nan and Leonard (Lou) Scott

Quasiprimitive "O'Nan-Scott Theorem"

- I proved it in 1991 to help study 2-arc transitive graphs
- I needed it to make normal quotient analysis powerful
- Both versions partition the family of finite (quasi)primitive permutation groups into useful subfamilies – provide extra information on each
- Recall: Primitive: maximal stabiliser subgroups - only trivial blocks
- Quasiprimitive: nontrivial normal subgroups transitive

O’Nan-Scott style Theorems

- The Perth subdivision(!) $G < \mathrm{Sym}(V)$ (quasi)primitive
  - 8 sub-families depending on:
  - Minimal normal subgroups N of G
  - How many N?   Abstract structure of N?   N-action on V?
  - Each G lies in exactly one sub-family
- Will describe sub-families or “O’Nan-Scott types” by answering some questions.
- Fact: G has either 1 or 2 minimal normal subgroups – never more than 2
- Fact: Each minimal normal subgroup N = direct product of k copies of some simple group T

Two minimal normal subgroups N and M

- N, M isomorphic, nonabelian, regular on V [only identity fixes a point]
- Identify V with N so that N acts by right multiplication n: x -> xn
  - M acts by left multiplication m: x -> m^{-1}x
  - G = N.Y where Inn(N) < Y < Aut(N)
  - [socle] NM = N.Inn(N);
  - Largest group Hol(N)=N.Aut(N) called holomorph of N
- N simple – HS type [Holomorph Simple]
- N not simple – HC type [Compound Holomorph]
- All quasiprimitive HS and HC groups are primitive

Now assume unique minimal normal subgroup N (the socle of G)

Cases:

- N abelian
- N nonabelian simple, N=T
- N nonabelian non-simple, N=T^k
  - OK so 2 + 3 =5 You said there were 8 types
  - We have to subdivide a bit further

N abelian

- N a product of isomorphic simple groups means
  - N= additive group of a finite vector space V
  - Can identify the point set with V and N acts by translations n : x -> x + n
- G = N Y with Y an irreducible subgroup of GL(V)
  - GL(V) is group of nonsingular linear transformations of V
  - Irreducible: only invariant subspaces are V and 0
- Largest group AGL(V) = N.GL(V) the affine group
  - Type called HA (Holomorph of Abelian group)
  - All quasiprimitive HA groups are primitive

N a nonabelian simple group T

- T < G <Aut(T) type AS [Almost Simple]
- G quasiprimitive: N=T transitive
- G primitive: Stabiliser maximal
  - Some quasiprimitive AS groups not primitive

$N = T \times T \times \ldots \times T = T^k$, $k > 1$

Cases depend on stabiliser N_v of point v

- $N_v = 1$
- N_v = diagonal subgroup* of N
- N_v = "product of diagonals"
- N_v contained in R^k for some proper subgroup R of T

- Each of these 4 types contains some non-primitives

- * Each diagonal subgroup of N conjugate in Aut(N) to
- $D := \{ (t, t, \ldots, t) \mid t \in T \}$ isomorphic to T

Twisted wreath type TW: $N_v = 1$

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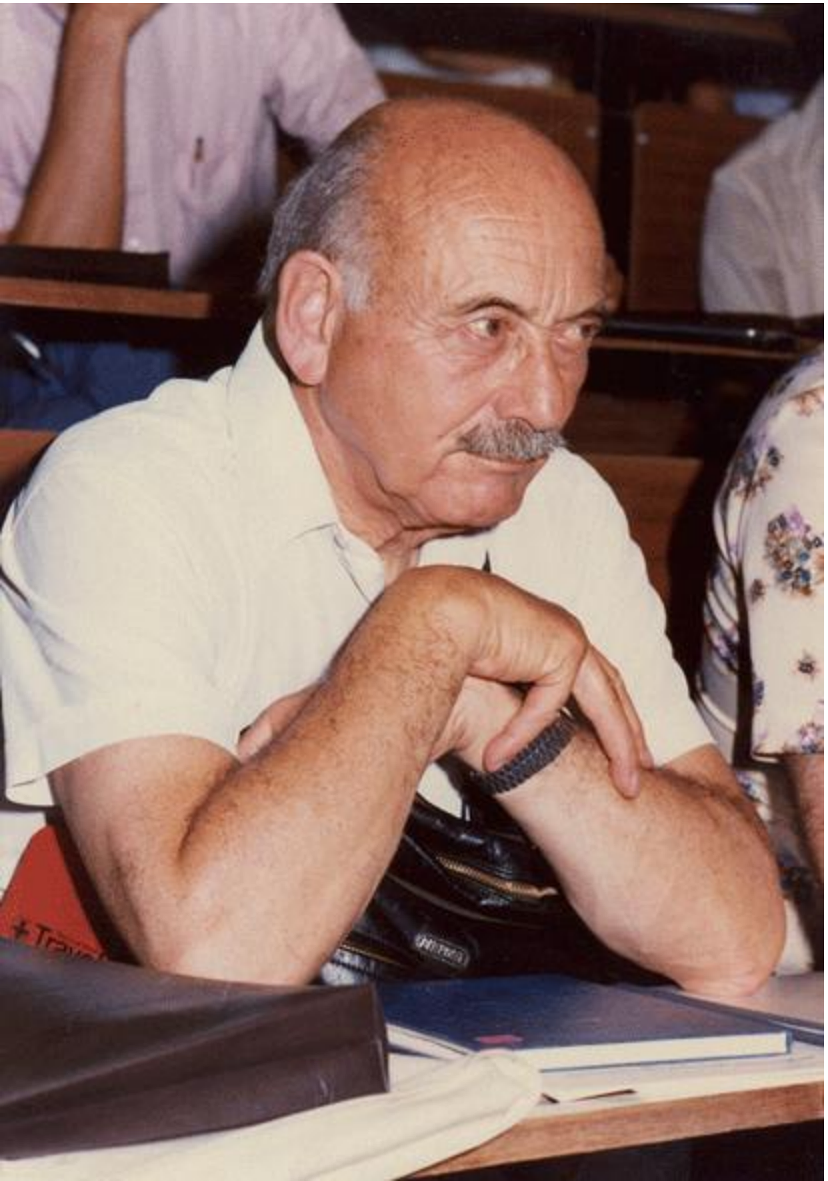

- $G = N\,Y, \quad Y < Aut(N)$
- Identify V with N
- Y faithful transitive action on k factors of $N = T \times T \times \ldots \times T$
- G called twisted wreath product of T and Y
- First studied (and named) by B H Neumann in 1930s
- Additional conditions needed to make G primitive

Simple Diagonal SD: N_v = diag subgroup

- Can identify V with product of first k-1 factors T of N
- Action can be described explicitly $|V|=|T|^{k-1}$
- $G= N G_v$ and $G_v < Aut(T) \times S_k$
- Largest SD group: stabiliser is $Aut(T) \times S_k$
- Quasiprimitive: G induces a transitive subgroup of S_k
- Primitive: G induces a primitive subgroup of S_k

Compound Diagonal CD:
Nv product of diagonals

- Wimping out [losing time]
- Construct CD group from an SD group in a similar way to constructing a PA group from an AS group (see next slide)
- Quasiprimitive CD group is primitive when the SD “component” is primitive

Product action

- Primitive case: $V = U \times U \times \cdots \times U = U^k$
- Group elements: products of two kinds of permutations

  $g = hs = (h_1, h_2, \ldots, h_k)s$ where

$$
\begin{array}{llll}
h: & (u_1, u_2, \ldots, u_k) & \to & (u_1^{h_1}, u_2^{h_2}, \ldots, u_k^{h_k}) \\
s: & (u_1, u_2, \ldots, u_k) & \to & (u_{1s^{-1}}, u_{2s^{-1}}, \ldots, u_{ks^{-1}})
\end{array}
$$

- h_i come from AS primitive group $H \leq \mathrm{Sym}(U)$,

  $T \leq H \leq \mathrm{Aut}(T)$; s comes from transitive $K \leq Sym(k)$
- Largest PA group (given H, K) is wreath product $H \wr K$ in 'product action' G contains T^k

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Conditions for Primitive PA

- AS primitive group $H \leq \mathrm{Sym}(U)$, $T \leq H \leq \mathrm{Aut}(T)$
- transitive $K \leq \mathrm{Sym}(k)$

- implies $W = H \wr K$ is primitive in product action.
- Each PA primitive group G 'induces' an AS primitive group H and transitive K and contains the socle $N = T^k$ of W
- N is transitive and for $(u, u, \ldots, u) \in V = U^k$, $N_{(u,u,\ldots,u)} = R^k$ where $R = T_u$

What about quasiprimitive PA groups?

- Each PA quasiprimitive group G on V determines
  - AS quasiprimitive group $H \leq \mathrm{Sym}(U)$, $T \leq H \leq \mathrm{Aut}(T)$
  - transitive $K \leq \mathrm{Sym}(k)$
- implies $W = H \wr K$ is quasiprimitive in product action on U^k but W is only a special kind of quasiprimitive group
- G-Block system B in V and $G^B \cong G$ permutationally isomorphic to a quasiprimitive action on U^k
- For point v in block labeled $(u, u, \ldots, u) \in B = U^k$, the stabiliser N_v is subdirect subgroup* of block stabiliser R^k (* projects onto R in each coordinate)

Powerful theory of (quasi)primitive groups

- Simple group theory – critically important
- Representation theory – affine cases

- Can we get “too much of a good thing”?
- Even for 2-arc-transitive graphs (Michael’s topic) we can get arbitrarily many pairwise non-isomorphic basic (quasiprimitive) normal quotient graphs
- Studying basic quotients helps us understand whole class – usually not classify them

Some studies demand richer Degen (F)

- F = { edge-transitive strongly regular graphs}
- Normal quotients either strongly regular or complete graphs
- Should we take Degen(F) = { K_n all n }
- What does a basic strongly regular graph look like?

F = { edge-transitive strongly regular graphs (srg)}

Degen(F) = { K_n all n } s

- Basic srg is
  - "Not complete but all proper normal quotients are complete"
  - If all normal quotients are just K1 get quasiprimitive srg – very nice
  - Other basic examples we called "quotient complete"

- Very interesting class (more generally than srg) – ask Carmen about them