Reachability relations, transitive digraphs and groups

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Introduction


Reachability relation on edges: $e$ is reachable from $f$ if there exists an alternating walk containing $e$ and $f$.

Reachability relation on vertices:
$W = (v_0, \epsilon_1, v_1, \epsilon_2, v_2, \ldots, \epsilon_n, v_n)$ from $v_0$ to $v_n$ is a sequence of $n + 1$ vertices and $n$ indicators $\epsilon_1, \ldots, \epsilon_n$ such that

$$
\epsilon_j = 1 \implies (v_{j-1}, v_j) \in E(W),
$$

$$
\epsilon_j = -1 \implies (v_j, v_{j-1}) \in E(W).
$$

Weight $\omega(W) = \sum_{i=1}^{n} \epsilon_i$
\[ uR_k^+ v \]

if there exists a walk \( W \) from \( u \) to \( v \) with \( \omega(W) = 0 \) and \( \omega(0 \cdot W_j) \in [0, k] \) for every \( 0 \leq j \leq |W| \). Analogously \( uR_k^- v \).

\[
R_k^+(v) = \{ u \in V(D) | vR_k^+ u \} \\
R_k^-(v) = \{ u \in V(D) | vR_k^- u \}
\]

\( R_k^+ \subseteq R_{k+1}^+ \), \( R_k^- \subseteq R_{k+1}^- \)

\[
R^+ = \bigcup_{k \in \mathbb{Z}^+} R_k^+ , \quad R^- = \bigcup_{k \in \mathbb{Z}^+} R_k^-
\]

\((R_k^+ )_{k \in \mathbb{Z}^+}, (R_k^- )_{k \in \mathbb{Z}^+}\)

exponent \( \exp^+(D) \) is the smallest nonnegative integer \( k \) such that \( R_k^+ = R^+ \). Analogously \( \exp^-(D) \).
$D$ is connected, vertex-transitive, infinite, locally finite

Structure of $D/R^+$:

- a finite cycle
- directed infinite line
- regular tree with indegree $1$ and outdegree $>1$


Are there connections between $R^+_k$ ($R^-_k$) and the end structure of $D$?

$D$ has property $Z$ if there exists a homomorphism from $D$ onto the directed infinite line.
If $D$ has infinitely many ends, then it has property $Z$ if and only if at least one of the sequences $(R^+_k)_{k \in \mathbb{Z}^+}$ and $(R^-_k)_{k \in \mathbb{Z}^+}$ is infinite.

If $D$ has property $Z$ and the sequences $(R^+_k)_{k \in \mathbb{Z}^+}$ and $(R^-_k)_{k \in \mathbb{Z}^+}$ are both finite and there exists an integer $k \geq 1$ such that $R^+_k$ (and hence $R^-_k$) has infinite equivalence classes, then $D$ has one end.

If $D$ has two ends, then it has property $Z$ if and only if for each integer $k \geq 1$ at least one (and hence both) of the relations $R^+_k$ and $R^-_k$ have finite equivalence classes.
Connections between $R_k^+$ ($R_k^-$) and growth properties?

$$f_D(v, n) = |\{ u \in V(D) | \text{dist}_D(v, u) \leq n \}|$$

- polynomial growth: $f_D(n) \leq cn^d$ for all $n \geq 1$
- exponential growth: $f_D \geq c^n$ for all $n \geq 1$
- intermediate growth: E. g. $2^{\sqrt{n}} < f_D(n) < 2^{n \log_{32}^{31}}$

If at least one of the sequences $(R_k^+)_k \in \mathbb{Z}^+$ and $(R_k^-)_k \in \mathbb{Z}^+$ is infinite, then $D$ has exponential growth.
Both sequences finite $\Rightarrow$ polynomial or intermediate growth

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- Is it possible to find conditions for $R_k^+(R_k^-)$ which imply polynomial or intermediate growth?
- Do there exist bounds for $\exp^+(D) (\exp^-(D))$ in the case of polynomial growth?
If an abelian group acts transitively on $D$, then 
$\exp^+ (D) = \exp^- (D) = 1$.

Nilpotent groups?

$G^0 = G$, $G^{i+1} = [G^0, G^i], i \geq 0$

\[ G = G^0 \triangleright G^1 \triangleright \ldots \triangleright G^k \triangleright G^{k+1} = 1 \]

nilpotent of class $k$.

Let $G$ be a nilpotent group of class $k \geq 0$ acting transitively on $D$. Then $\exp^+ (D) \leq k + 1$ and $\exp^- (D) \leq k + 1$.
This bound is tight! \( \rightarrow D_8 \)

Infinite family of nilpotent groups:

\( G_n \) semidirect product of the elementary abelian group \( \mathbb{Z}_{2^n} \) by the cyclic group \( \mathbb{Z}_{2^{n-1}} \) generated by \( G_n = \langle f, a_1, a_2, \ldots, a_n \rangle \). \( f \) cyclic of order \( 2^{n-1} \), \( a_i \) involutions. \( fa_i f^{-1} = a_i a_{i+1}, 1 \leq i \leq n - 1 \). \( a_i a_j = a_j a_i, fa_n = a_n f \).

\( S = \{ f, fa_1 \}, \langle S^{-i} S^i \rangle = \langle a_1, a_2, \ldots, a_i \rangle, 1 \leq i \leq n. \Rightarrow \exp^{-}(Cay(G_n, S)) = n. \)

\( G_n \) is nilpotent of class \( n - 1 \). \( G^{(i)} = \langle a_{i+1}, a_{i+2}, \ldots, a_n \rangle \) holds for each \( i, 1 \leq i \leq n - 1 \). Also \( G^{(n)} = 1. \)
No bound for solvable groups! → lamplighter group. $L$ is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$

$$L = \langle a, t | a^2, [t^m at^{-m}, t^n at^{-n}], m, n \in \mathbb{Z} \rangle.$$

$S = \{t, at\}$, $\text{Cay}(L, S)$ horocyclic product of two trees with indegree 1, outdegree 2.
\( R_k^+ (R_k^-) \) and growth of groups

\[ G \text{ finitely generated with polynomial growth} \Rightarrow G \text{ contains a normal nilpotent subgroup } N \text{ of finite index.} \]

- Let the finitely generated group \( G \) act transitively on the connected digraph \( D \) such that a normal nilpotent subgroup \( N \) of \( G \), where \( N \) is nilpotent of class \( k \geq 0 \), acts with \( m, 1 \leq m < \infty \), orbits on \( D \). Then
  \[ \exp^+(D) \leq m(k + 1) + m - 1 \text{ and } \exp^-(D) \leq m(k + 1) + m - 1. \]

All examples we know satisfy \( \exp^+(D) \leq m(k + 1) \) and \( \exp^-(D) \leq m(k + 1) \).

- Let the finitely generated group \( G \) act transitively on the connected digraph \( D \) such that a normal abelian subgroup \( N \) of \( G \) acts with \( m, 1 \leq m < \infty \), orbits on \( D \). Then
  \[ \exp^+(D) \leq m \text{ and } \exp^-(D) \leq m. \]
The orders of the finite subgroups of $\text{GL}(n, \mathbb{Z})$ are bounded by some function $g(n)$ alone.

Let $G$ be a finitely generated torsion-free group with polynomial growth of degree $d$. Then $G$ contains a normal nilpotent subgroup of class $< \sqrt{2d}$ and index at most $g(d)$, where $g(d)$ is the above function.

Let $G$ be a finitely generated torsion-free group with polynomial growth of degree $d$. Then for any Cayley graph $D$ of $G$, $\exp^+(D) \leq g(d)(\sqrt{2d} + 1) + g(d) - 1$ and $\exp^-(D) \leq g(d)(\sqrt{2d} + 1) + g(d) - 1$. 
Is it true that every finitely generated infinite simple group has exponential growth? (Grigorchuk)

- If a finitely generated infinite simple group $G$ does not have exponential growth, then for every finite generating set $S$ of $G$ there is a finite integer $k_S \geq 1$, such that $R_{k_S}^+ = R_{k_S}^-$ is universal in $C(G, S)$.
- Let $G$ be a finitely generated infinite simple group and let $S$ denote a finite generating set. Furthermore, let $H \subseteq G$ denote the set of all those $h \in G$ which leave invariant at least one equivalence class of $R_1^+$ on $C(G, S)$. Then $\langle H \rangle = G$. 