A classification of sharp tridiagonal pairs

Tatsuro Ito   Kazumasa Nomura   Paul Terwilliger
This talk concerns a linear algebraic object called a \textit{tridiagonal pair}.

We will describe its features such as the eigenvalues, dual eigenvalues, shape, tridiagonal relations, split decomposition and parameter array.

We will then define an algebra \( T \) by generators and relations, and prove a theorem about its structure called the \textit{\( \mu \)-Theorem}.

We will use the \textit{\( \mu \)-Theorem} to obtain a \textit{Classification Theorem} for sharp tridiagonal pairs.
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We will then define an algebra $\mathbb{T}$ by generators and relations, and prove a theorem about its structure called the $\mu$-**Theorem**.

We will use the $\mu$-Theorem to obtain a **Classification Theorem** for sharp tridiagonal pairs.
We recall the notion of a Leonard pair. To do this, we first recall what it means for a matrix to be \textit{tridiagonal}.

The following matrices are tridiagonal:

\[
\begin{pmatrix}
2 & 3 & 0 & 0 \\
1 & 4 & 2 & 0 \\
0 & 5 & 3 & 3 \\
0 & 0 & 3 & 0
\end{pmatrix}, \quad \begin{pmatrix}
2 & 3 & 0 & 0 \\
0 & 4 & 2 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 1 & 5
\end{pmatrix}
\]

Tridiagonal means each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. The tridiagonal matrix on the left is \textit{irreducible}. This means each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.
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Leonard pairs

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The Definition of a Leonard Pair

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**Definition**

Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. By a **Leonard pair** on $V$, we mean a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$ which satisfy both conditions below.

1. There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

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In a Leonard pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (1), (2) above.
Example of a Leonard pair

For any integer $d \geq 0$ the pair

$$A = \begin{pmatrix} 0 & d & 0 & \cdots & 0 \\ 1 & 0 & d-1 & \cdots & 0 \\ 2 & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & \cdots & \cdots & 1 \\ 0 & 0 & d & 0 \end{pmatrix},$$

$$A^* = \text{diag}(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbb{F}^{d+1}$, provided the characteristic of $\mathbb{F}$ is 0 or an odd prime greater than $d$.
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\cdot & \cdot & \cdot & \cdot \\
0 & \cdot & 1 & d & 0
\end{pmatrix},
\]

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A^* = \text{diag}(d, d-2, d-4, \ldots, -d)
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is a Leonard pair on the vector space $\mathbb{F}^{d+1}$, provided the characteristic of $\mathbb{F}$ is 0 or an odd prime greater than $d$.

Reason: There exists an invertible matrix $P$ such that $P^{-1}AP = A^*$ and $P^2 = 2^d I$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger
There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the following types:

$q$-Racah,
$q$-Hahn,
dual $q$-Hahn,
$q$-Krawtchouk,
dual $q$-Krawtchouk,
quantum $q$-Krawtchouk,
affine $q$-Krawtchouk,
Racah,
Hahn,
dual-Hahn,
Krawtchouk,
Bannai/Ito,
orphans ($\text{char}(\mathbb{F}) = 2$ only).

This family coincides with the terminating branch of the Askey scheme of orthogonal polynomials.
The theory of Leonard pairs is summarized in

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As before, we consider a pair of linear transformations $A : V \rightarrow V$ and $A^* : V \rightarrow V$. 
Definition of a Tridiagonal pair

We say the pair $A, A^*$ is a TD pair on $V$ whenever (1)–(4) hold below.

1. Each of $A, A^*$ is diagonalizable on $V$.
2. There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0, V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0, V_{\delta+1}^* = 0$.
4. There is no subspace $W \subseteq V$ such that $AW \subseteq W$ and $A^*W \subseteq W$ and $W \neq 0$ and $W \neq V$.
Referring to our definition of a TD pair, it turns out $d = \delta$; we call this common value the **diameter** of the pair.
We mentioned that a tridiagonal pair is a generalization of a Leonard pair.
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A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces $V_i$ and $V_i^*$ all have dimension 1.
The concept of a TD pair originated in \textit{algebraic graph theory}, or more precisely, the theory of \textit{Q-polynomial distance-regular graphs}. See

When working with a TD pair, it is helpful to consider a closely related object called a **TD system**.

We will define a TD system over the next few slides.
Referring to our definition of a TD pair,

An ordering \( \{ V_i \}_{i=0}^d \) of the eigenspaces of \( A \) is called standard whenever

\[
A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),
\]

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An ordering \( \{ V_i \}_{i=0}^d \) of the eigenspaces of \( A \) is called \textbf{standard} whenever

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In this case, the ordering \( \{ V_{d-i} \}_{i=0}^d \) is also standard and no further ordering is standard.
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An ordering \( \{ V_i \}_{i=0}^d \) of the eigenspaces of \( A \) is called **standard** whenever

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In this case, the ordering \( \{ V_{d-i} \}_{i=0}^d \) is also standard and no further ordering is standard.

A similar discussion applies to \( A^* \).
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In other words $E - I$ vanishes on the eigenspace and $E$ vanishes on all the other eigenspaces.
Definition

By a **TD system** on $V$ we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies the following:

1. $A, A^*$ is a TD pair on $V$.
2. $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A$.
3. $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of $A^*$.

Until further notice we fix a TD system $\Phi$ as above.
For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with the eigenspace $E_i V$ (resp. $E_i^* V$).
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We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the **eigenvalue sequence** (resp. **dual eigenvalue sequence**) of $\Phi$. 
A three-term recurrence

Theorem (Ito+Tanabe+T, 2001)

The expressions

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i},
\]

are equal and independent of i for \(2 \leq i \leq d - 1\).

Let \(\beta + 1\) denote the common value of the above expressions.
For the above recurrence the “simplest” solution is

\[ \theta_i = d - 2i \quad (0 \leq i \leq d), \]
\[ \theta^*_i = d - 2i \quad (0 \leq i \leq d). \]

In this case \( \beta = 2. \)

For this solution our TD system \( \Phi \) is said to have \textbf{Krawtchouk type}.
Solving the recurrence, cont.

For the above recurrence another solution is

\[ \theta_i = q^{d-2i} \quad (0 \leq i \leq d), \]
\[ \theta^*_i = q^{d-2i} \quad (0 \leq i \leq d), \]
\[ q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1. \]

In this case \( \beta = q^2 + q^{-2} \).

For this solution \( \Phi \) is said to have \( q \)-Krawtchouk type.
For the above recurrence the “most general” solution is

\[
\theta_i = a + bq^{2i-d} + cq^{d-2i} \quad (0 \leq i \leq d),
\]
\[
\theta_i^* = a^* + b^* q^{2i-d} + c^* q^{d-2i} \quad (0 \leq i \leq d),
\]
\[q, \ a, \ b, \ c, \ a^*, \ b^*, \ c^* \in \overline{\mathbb{F}},\]
\[q \neq 0, \ q^2 \neq 1, \ q^2 \neq -1, \ bb^* cc^* \neq 0.\]

In this case \(\beta = q^2 + q^{-2}\).

For this solution \(\Phi\) is said to have \textbf{q-Racah type}.
Some notation

For later use we define some polynomials in an indeterminate \( \lambda \).

For \( 0 \leq i \leq d \),

\[
\begin{align*}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\
\tau^*_i &= (\lambda - \theta^*_0)(\lambda - \theta^*_1) \cdots (\lambda - \theta^*_{i-1}), \\
\eta^*_i &= (\lambda - \theta^*_d)(\lambda - \theta^*_d_{-1}) \cdots (\lambda - \theta^*_d_{-i+1}).
\end{align*}
\]

Note that each of \( \tau_i, \eta_i, \tau^*_i, \eta^*_i \) is monic with degree \( i \).
It is known that for $0 \leq i \leq d$ the eigenspaces $E_i V$, $E_i^* V$ have the same dimension; we denote this common dimension by $\rho_i$.

**Lemma (Ito+Tanabe+T, 2001)**

The sequence $\{\rho_i\}_{i=0}^{d}$ is **symmetric** and **unimodal**; that is

\[
\rho_i = \rho_{d-i} \quad (0 \leq i \leq d), \\
\rho_{i-1} \leq \rho_i \quad (1 \leq i \leq d/2).
\]

We call the sequence $\{\rho_i\}_{i=0}^{d}$ the **shape** of $\Phi$. 
The shape \( \{ \rho_i \}_{i=0}^{d} \) of \( \Phi \) satisfies \( \rho_i \leq \rho_0 \binom{d}{i} \) \((0 \leq i \leq d)\).
What are the possible values for \( \rho_0 \)?
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We will explain this after a few slides.
Some relations

**Lemma**

Our TD system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ satisfies the following relations:

$$E_i E_j = \delta_{i,j} E_i, \quad E_i^* E_j^* = \delta_{i,j} E_i^*, \quad 0 \leq i, j \leq d,$$

$$1 = \sum_{i=0}^d E_i, \quad 1 = \sum_{i=0}^d E_i^*,$$

$$A = \sum_{i=0}^d \theta_i E_i, \quad A^* = \sum_{i=0}^d \theta_i^* E_i^*,$$

$$E_i^* A^k E_j^* = 0 \quad \text{if} \quad k < |i - j| \quad 0 \leq i, j, k \leq d,$$

$$E_i A^*^k E_j = 0 \quad \text{if} \quad k < |i - j| \quad 0 \leq i, j, k \leq d.$$

We call these last two equations the **triple product relations**.
Given the relations on the previous slide, it is natural to consider the algebra generated by $A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d$. We call this algebra $T$. Consider the space $E^*_0T^*_0$. Observe that $E^*_0T^*_0$ is an $F$-algebra with multiplicative identity $E^*_0$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger  
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Observe that $E_0^* T E_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity $E_0^*$. 
The algebra $E_0^* TE_0^*$

Theorem (Ito+Nomura+T, 2007)

(i) The $F$-algebra $E_0^* TE_0^*$ is commutative and generated by

$$E_0^* A^i E_0^* \quad 1 \leq i \leq d.$$ 

(ii) $E_0^* TE_0^*$ has no zero-divisors; in other words it is a field.

(iii) Viewing this field as a field extension of $F$, the index is $\rho_0$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger  

A classification of sharp tridiagonal pairs
The parameter $\rho_0$

Corollary (Ito+Nomura+T, 2007)

If $\mathbb{F}$ is algebraically closed then $\rho_0 = 1$.

We now consider some more relations in $T$. 
The tridiagonal relations

**Theorem (Ito+Tanabe+T, 2001)**

For our TD system $\Phi$ there exist scalars $\gamma, \gamma^*, \varrho, \varrho^*$ in $\mathbb{F}$ such that

\[
\]

\[
\]

The above equations are called the **tridiagonal relations**.
In the Krawtchouk case the tridiagonal relations become the **Dolan-Grady relations**

\[
\]

\[
\]

Here \( [r, s] = rs - sr \).
The $q$-Serre relations

In the $q$-Krawtchouk case the tridiagonal relations become the cubic $q$-Serre relations

\[ A^{*3} A - [3]_q A^{*2} A A^* + [3]_q A^* A A^*^2 - A A^{*3} = 0. \]

\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n = 0, 1, 2, \ldots \]
The sharp case

At this point it is convenient to make an assumption about our TD system $\Phi$. If the ground field $F$ is algebraically closed then $\Phi$ is sharp. Until further notice assume $\Phi$ is sharp.
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$\Phi$ is called **sharp** whenever $\rho_0 = 1$, where $\{\rho_i\}_{i=0}^{d}$ is the shape of $\Phi$. 

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The split decomposition

For $0 \leq i \leq d$ define

$$U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V).$$

It is known that

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)},$$

and for $0 \leq i \leq d$ both

$$U_0 + \cdots + U_i = E_0^* V + \cdots + E_i^* V,$$

$$U_i + \cdots + U_d = E_i V + \cdots + E_d V.$$

We call the sequence $\{U_i\}_{i=0}^d$ the split decomposition of $V$ with respect to $\Phi$. 
Theorem (Ito+Tanabe+T, 2001)

For $0 \leq i \leq d$ both

$$(A - \theta_i I)U_i \subseteq U_{i+1},$$
$$(A^* - \theta_i^* I)U_i \subseteq U_{i-1},$$

where $U_{-1} = 0$, $U_{d+1} = 0$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger

A classification of sharp tridiagonal pairs
Observe that for $0 \leq i \leq d$,

$$(A - \theta_{i-1}I) \cdots (A - \theta_1 I)(A - \theta_0 I)U_0 \subseteq U_i,$$

$$(A^* - \theta_1^* I) \cdots (A^* - \theta_{i-1}^* I)(A^* - \theta_i^* I)U_i \subseteq U_0.$$  

Therefore $U_0$ is invariant under

$$(A^* - \theta_1^* I) \cdots (A^* - \theta_i^* I)(A - \theta_{i-1} I) \cdots (A - \theta_0 I).$$

Let $\zeta_i$ denote the corresponding eigenvalue and note that $\zeta_0 = 1$.

We call the sequence $\{\zeta_i\}_{i=0}^d$ the split sequence of $\Phi$. 

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Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger  
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The split sequence $\{\zeta_i\}_{i=0}^d$ is characterized as follows.

**Lemma (Nomura+T, 2007)**

For $0 \leq i \leq d$,

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$
A restriction on the split sequence

The split sequence \( \{\zeta_i\}_{i=0}^d \) satisfies two inequalities.

**Lemma (Ito+Tanabe+T, 2001)**

\[
0 \neq E_0^* E_d E_0^*, \\
0 \neq E_0^* E_0 E_0^*.
\]

Consequently

\[
0 \neq \zeta_d, \\
0 \neq \sum_{i=0}^d \eta_{d-i}(\theta_0) \eta_{d-i}^*(\theta_0^*) \zeta_i.
\]
The TD system $\Phi$ is determined up to isomorphism by the sequence

$$\left(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d\right).$$

We call this sequence the parameter array of $\Phi$. 

**Lemma (Ito+ Nomura+T, 2008)**

*The TD system $\Phi$ is determined up to isomorphism by the sequence*

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$\mathbb{T}$ is an abstract version of $T$ defined by generators and relations. We will define $\mathbb{T}$ shortly.
Feasible sequences

Definition

Let $d$ denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$. This sequence is called **feasible** whenever both

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$);

(ii) the expressions $\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}$, $\frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$ are equal and independent of $i$ for $2 \leq i \leq d - 1$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger

A classification of sharp tridiagonal pairs
The algebra $\mathbb{T}$

**Definition**

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Let $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ denote the $\mathbb{F}$-algebra defined by generators $a$, $\{e_i\}_{i=0}^d$, $a^*$, $\{e_i^*\}_{i=0}^d$ and relations

$$e_i e_j = \delta_{i,j} e_i, \quad e_i^* e_j^* = \delta_{i,j} e_i^*, \quad 0 \leq i, j \leq d,$$

$$1 = \sum_{i=0}^d e_i, \quad 1 = \sum_{i=0}^d e_i^*,$$

$$a = \sum_{i=0}^d \theta_i e_i, \quad a^* = \sum_{i=0}^d \theta_i^* e_i^*,$$

$$e_i^* a^k e_j^* = 0 \quad \text{if} \quad k < |i - j| \quad 0 \leq i, j, k \leq d,$$

$$e_i a^k e_j = 0 \quad \text{if} \quad k < |i - j| \quad 0 \leq i, j, k \leq d.$$
Over the next few slides, we explain how TD systems are related to finite-dimensional irreducible $\mathbb{T}$-modules.
Lemma

Let \((A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})\) denote a TD system on \(V\) with eigenvalue sequence \(\{\theta_i\}_{i=0}^{d}\) and dual eigenvalue sequence \(\{\theta_i^*\}_{i=0}^{d}\). Let \(T = T(p, \mathbb{F})\) where \(p = (\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d})\). Then there exists a unique \(T\)-module structure on \(V\) such that \(a, e_i, a^*, e_i^*\) acts as \(A, E_i, A^*, E_i^*\) respectively. This \(T\)-module is irreducible.
Lemma

Fix a feasible sequence $p = (\{\theta_i\}_{i=0}^{d}; \{\theta^*_i\}_{i=0}^{d})$ and write $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$. Let $V$ denote a finite-dimensional irreducible $\mathbb{T}$-module.

(i) There exist nonnegative integers $r$, $\delta$ ($r + \delta \leq d$) such that for $0 \leq i \leq d$,

$$e^*_i V \neq 0 \text{ if and only if } r \leq i \leq r + \delta.$$ 

(ii) There exist nonnegative integers $t$, $\delta^*$ ($t + \delta^* \leq d$) such that for $0 \leq i \leq d$,

$$e_i V \neq 0 \text{ if and only if } t \leq i \leq t + \delta^*.$$ 

(iii) $\delta = \delta^*$.

(iv) The sequence $(a; \{e_i\}_{i=t}^{t+\delta}; a^*; \{e_i^*\}_{i=r}^{r+\delta})$ acts on $V$ as a TD system of diameter $\delta$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger
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The structure of $\mathbb{T}$
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As we did with $T$ we consider the space $e_0^* \mathbb{T} e_0^*$. 
Fix a feasible sequence \( p = (\{\theta_i\}_{i=0}^{d}; \{\theta_i^*\}_{i=0}^{d}) \) and consider the \( \mathbb{F} \)-algebra \( \mathbb{T} = \mathbb{T}(p, \mathbb{F}) \).

As we did with \( T \) we consider the space \( e_0^* \mathbb{T} e_0^* \).

Observe that \( e_0^* \mathbb{T} e_0^* \) is an \( \mathbb{F} \)-algebra with multiplicative identity \( e_0^* \).
Notation

Let \( \{ \lambda_i \}_{i=1}^d \) denote mutually commuting indeterminates.

Let \( \mathbb{F}[\lambda_1, \ldots, \lambda_d] \) denote the \( \mathbb{F} \)-algebra consisting of the polynomials in \( \{ \lambda_i \}_{i=1}^d \) that have all coefficients in \( \mathbb{F} \).
Theorem (Ito+Nomura+T, 2009)

There exists an $\mathbb{F}$-algebra isomorphism

$$\mathbb{F}[\lambda_1, \ldots, \lambda_d] \rightarrow e_0^* T e_0^*$$

that sends

$$\lambda_i \mapsto e_0^* a^i e_0^*$$

for $1 \leq i \leq d$. 
Proof summary: We first verify the result assuming $p$ has $q$-Racah type. To do this we make use of the quantum affine algebra $U_q(\hat{sl}_2)$. We identify two elements in $U_q(\hat{sl}_2)$ that satisfy the tridiagonal relations. We let these elements act on $U_q(\hat{sl}_2)$-modules of the form $W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where each $W_i$ is an evaluation module of dimension 2. Each of these actions gives a TD system of $q$-Racah type which in turn yields a $\mathbb{T}$-module. The resulting supply of $\mathbb{T}$-modules is sufficiently rich to contradict the existence of an algebraic relation among $\{e_0^* a^i e_0^*\}_{i=1}^d$.

We then remove the assumption that $p$ has $q$-Racah type. In this step the main ingredient is to show that for any polynomial $h$ over $\mathbb{F}$ in $2d + 2$ variables, if $h(p) = 0$ under the assumption that $p$ is $q$-Racah, then $h(p) = 0$ without the assumption.
A classification of sharp tridiagonal systems

Theorem (Ito+Nomura+T, 2009)

Let \( \{\theta_i\}_{i=0}^{d}; \{\theta^*_i\}_{i=0}^{d}; \{\zeta_i\}_{i=0}^{d} \) (1) denote a sequence of scalars in \( \mathbb{F} \). Then there exists a sharp TD system \( \Phi \) over \( \mathbb{F} \) with parameter array (1) if and only if:

(i) \( \theta_i \neq \theta_j, \theta^*_i \neq \theta^*_j \) if \( i \neq j \) \( (0 \leq i, j \leq d) \);

(ii) the expressions \( \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \) are equal and independent of \( i \) for \( 2 \leq i \leq d - 1 \);

(iii) \( \zeta_0 = 1, \zeta_d \neq 0 \), and

\[
0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.
\]

Suppose (i)–(iii) hold. Then \( \Phi \) is unique up to isomorphism of TD systems.
The classification: proof summary

Proof ("only if"): By our previous remarks.

Proof ("if"): Consider the algebra $\mathbb{T} = \mathbb{T}(p, \mathbb{F})$ where $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$.

By the $\mu$-Theorem $e_0^*\mathbb{T}e_0^*$ is a polynomial algebra.

Therefore $e_0^*\mathbb{T}e_0^*$ has a 1-dimensional module on which

$$e_0^*\tau_i(a)e_0^* = \frac{\zeta_i e_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

for $1 \leq i \leq d$.

The above 1-dimensional $e_0^*\mathbb{T}e_0^*$-module induces a $\mathbb{T}$-module $V$ which turns out to be finite-dimensional; by construction $e_0^*V$ has dimension 1.

One checks that the $\mathbb{T}$-module $V$ has a unique maximal proper submodule $M$. 

Tatsuro Ito, Kazumasa Nomura, Paul Terwilliger

A classification of sharp tridiagonal pairs
Consider the irreducible $\mathbb{T}$-module $V/M$.

By the inequalities in (iii),

$$e_0^* e_d e_0^* \neq 0, \quad e_0^* e_0 e_0^* \neq 0$$

on $V/M$.

Therefore each of $e_0, e_d$ is nonzero on $V/M$.

Now the $\mathbb{T}$-generators $(a; \{ e_i \}_{i=0}^d; a^*; \{ e_i^* \}_{i=0}^d)$ act on $V/M$ as a sharp TD system of diameter $d$.

One checks that this TD system has the desired parameter array $(\{ \theta_i \}_{i=0}^d; \{ \theta_i^* \}_{i=0}^d; \{ \zeta_i \}_{i=0}^d)$.

QED
Summary

We defined a TD system and discussed its eigenvalues, dual eigenvalues, shape, tridiagonal relations, split decomposition and parameter array.

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THE END