The schurity problem for quasi-thin association schemes

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Coherent configurations (D. Higman, 1970)

A pair $X = (\Omega, S)$ where $\Omega$ is a finite set, and $S$ is a partition of $\Omega \times \Omega$, is called a coherent configuration if:

1. $\Omega = \{(\alpha, \alpha) : \alpha \in \Omega\}$ belongs to the set $S \cup$ of all unions of the relations from $S$,.
2. $S$ contains $s^* = \{(\alpha, \beta) : (\beta, \alpha) \in s\}$ for all $s \in S$,
3. for all $r, s, t \in S$ the intersection number $c_{rs} = \{|\{(\beta \in \Omega : (\alpha, \beta) \in r, (\beta, \gamma) \in s\}|$ does not depend on the choice of $(\alpha, \gamma) \in t$.

The numbers $|\Omega|$ and $|S|$ are the degree and rank of $X$; when $\Omega \in S$ the coherent configuration $X$ is called homogeneous or association scheme.
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Schurian coherent configurations

A permutation group $G \leq \text{Sym}(\Omega)$ acts naturally on $\Omega \times \Omega$:

$$(\alpha, \beta) g := (\alpha g, \beta g), \quad \alpha, \beta \in \Omega, \quad g \in G.$$ 

Set $X = (\Omega, S)$ where $S = \text{Orb}(G, \Omega \times \Omega)$.

Proposition

1. $X$ is a coherent configuration,
2. $S \cup \Omega$ is the set of all $G$-invariant binary relations,
3. $X$ is a scheme iff $G$ is transitive.

We say that $X$ is the coherent configuration of the group $G$.

Definition

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The schurity problem

Problem: characterize all schurian coherent configurations belonging to a given class.

The smallest degree of a non-schurian scheme is 15 (the Hanaki-Miamoto list), and coherent configuration $\geq 8$ (A.Leman, 1970).

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2. The Tits theory on spherical buildings solves the schurity problem in a class of the Coxeter schemes (Z, 2005);
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**Definition**

The scheme $\mathcal{X}$ is **thin** if $S = S_1$, and **quasi-thin** if $S = S_1 \cup S_2$ where $S_i$ is the set of all $s \in S$ with $n_s = i$. 

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Theorem (B. Weisfeiler, 1975)

1. A thin scheme is Schurian ($G = S_1$),
2. A primitive quasi-thin scheme is Schurian ($G \in \{C_p, D_{2p}\}$).

In a quasi-thin scheme given $s \in S_2$ there is a unique $s^\perp \in S$ for which $c_{s^\perp s}^s \neq 0$; the relation $s^\perp$ is the orthogonal of $s$; the set of all orthogonals is denoted by $S^\perp$.

Theorem (H-M, 2002 and Z-M, 2008)

A quasi-thin scheme is Schurian whenever $|S^\perp| = 1$ and $S^\perp \subset S_1$, or $S^\perp \subset S_2$. 


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A quasi-thin scheme $X = (\Omega, S)$ is called a Klein scheme if the set $\{ 1 \cup S \} \perp S$ is a Klein subgroup of the group $S$ (elementary abelian group of order 4); the number $|\Omega|/|S|$ is the index of $X$.

Any non-schurian quasi-thin scheme of degree $n \in \{16, 28, 32\}$ is a Klein scheme of index 4 (for $n = 16, 32$), or index 7.

**Main Theorem**

1. Any schurian quasi-thin scheme of degree $n$ is a scheme of a permutation group of order $n$ or $2n$;
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Corollaries

A scheme is commutative if $ct = tcr$; commutative thin schemes are in $1-1$-correspondence with finite abelian groups.

Corollary
Any commutative quasi-thin scheme is schurian.

The proof of the Main Theorem is based on the technique developed in [M-P, 2009] to apply to the schurity problem for pseudocyclic schemes.

As a byproduct of the proof one can get the following result.

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The proof of the Main Theorem is based on the technique developed in [M-P, 2009] to apply to the schurity problem for pseudocyclic schemes. As a byproduct of the proof one can get the following result.

**Theorem**

Any non-Kleinian quasi-thin scheme is uniquely determined by its array of intersection numbers.