Marko Orel

Rogla, 18.5.2013

- Preserver problems
- 2 Adjacency preservers
- Some techniques related to other mathematical areas
- Special theory of relativity
- 6 Hamiltonicity

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Example 1: Determinant preservers

Frobenius, 1897

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$$m = p_1 p_2 \cdots p_k$$
 product of distinct odd primes $\mathbb{Z}_m = \{0, 1, 2, \dots, m-1\}$

$$e \in \mathbb{Z}_m$$
 is idempotent if $e^2 = e$
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A linear map $\Phi: M_n(\mathbb{Z}_m) \to M_n(\mathbb{Z}_m)$ preserves idempotents, that is, $\Phi(A)^2 = \Phi(A)$ whenever $A^2 = A$, if and only if

$$\Phi(A) = eP(fA + (1-f)A^{\top})P^{-1},$$

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$$\mathcal{M} = a$$
 set of matrices

$$A, B \in \mathcal{M}$$
 are *adjacent* if $rk(A - B)$ is minimal and nonzero

$$\mathfrak{M} \in \{M_{m \times n}(\mathbb{F}), S_n(\mathbb{F}), H_n(\mathbb{F})\} \Longrightarrow \mathrm{rk}(A-B) = 1$$

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 $\Phi: \mathcal{M} \rightarrow \mathcal{M}$ preserves adjacency in both directions, if

A,B are adjacent $\Longrightarrow \Phi(A),\Phi(B)$ are adjacent.

$$\Gamma = (V, E), V = M, E = \{\{A, B\} : A \text{ and } B \text{ are adjacent}\}$$

Bijective adj. preserves in both directions = automorphisms of Γ

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Bijective maps that preserves adjacency in both directions or

$$\mathcal{M} \in \{M_{m \times n}(\mathbb{F}), S_n(\mathbb{F}), H_n(\mathbb{F}), A_n(\mathbb{F})\}$$

are characterized by Hua's fundamental theorem of geometry of **** matrices.

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Some generalizations

- $H_n(\mathbb{C})$ (Huang, Šemrl 2008)
- $S_n(\mathbb{R})$ (Legiša 2011)
- $H_n(\mathbb{F}_{q^2})$ (Orel 2009)
- $S_n(\mathbb{F}_q)$, $n \geq 3$ (Orel 2012)
- $M_{m \times n}(\mathbb{D})$, some additional assumptions (Šemrl, accepted)
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- Graph theory (eigenvalues, chromatic number)
- Geometry ((non)existence of ovoids/spreads in hermitian polar spaces)

Graph Γ is a *core* if $\operatorname{Aut}(\Gamma) = \operatorname{End}(\Gamma)$

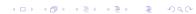
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 $S_n(\mathbb{F}_q), n \ge 3$
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Godsil, Royle, 2011

If Γ connected regular, $\operatorname{Aut}(\Gamma)$ acts transitively on pairs of vertices at distance 2, then Γ is a core or $\chi(\Gamma) = \omega(\Gamma)$

$$\chi(\Gamma) \ge 1 + \frac{\lambda_{\mathsf{max}}}{-\lambda_{\mathsf{min}}}$$



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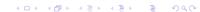
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Special theory of relativity

A Minkowski space-time M_4 is \mathbb{R}^4 equipped with a product

$$(\mathbf{r}_1, \mathbf{r}_2) = -x_1x_2 - y_1y_2 - z_1z_2 + c^2t_1t_2$$

between events $\mathbf{r}_1 := (x_1, y_1, z_1, ct_1)$ and $\mathbf{r}_2 := (x_2, y_2, z_2, ct_2)$.

A map $\phi: M_4 \to M_4$ preserves the speed of light if

$$(\phi(\mathbf{r}_1) - \phi(\mathbf{r}_2), \phi(\mathbf{r}_1) - \phi(\mathbf{r}_2)) = 0$$
 whenever $(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_2) = 0$.

These maps are closely related to adjacency preservers on 2×2 hermitian matrices.



There are only 5 known connected vertex transitive graphs that are not hamiltonian: K_2 , Petersen graph, Coxeter graph, two graphs derived from Petersen/Coxeter graph

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 $HGL_n(\mathbb{F}_4)$ vertex transitive $SGL_n(\mathbb{F}_2)$ vertex transitive for odd n

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