Counting cycles in vertex-transitive graphs

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$$\mathcal{DM} = 60$$

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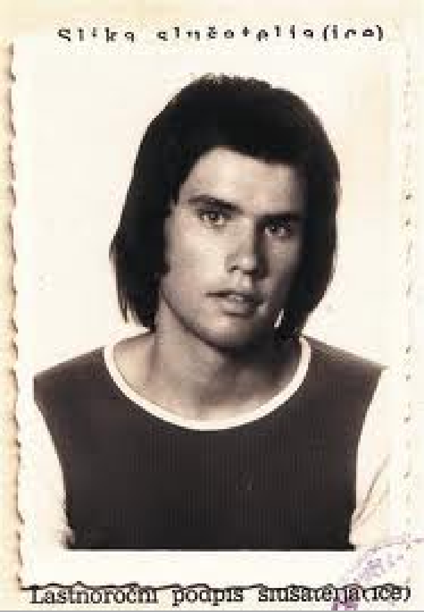
The only picture of me and Dragan


Slika slušatelja(ice)
Lastnoročni podpis slušatelja(ice)

Prehistory

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- Cayley graphs $G = C(\Gamma, X)$:
$V(G) = \Gamma, \quad E(G) = \{ \{a, ax\} \mid a \in \Gamma, x \in X \}$

$$\Rightarrow$$

Cayley graphs exist for all orders $n \geq 1$

Dragan Marušič:

Classify the orders n for which there exists a non-Cayley vertex-transitive graph (**VTNCG**) of order n; the **non-Cayley numbers**.

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Theorem (Fronček, Rosa, Širáň)

Let $G = C(\Gamma, X)$ and p be an odd prime. Then the number of closed oriented walks of length p based at any vertex of G is congruent modulo p to the number of generators in X of order p.

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Definition

The **coset graph** $G = Cos(\Gamma, H, X)$, $H \leq G$, $X \subseteq G$, $H \cap X =:$

$$V(G) = \{\ aH \mid a \in \Gamma\ \}, \quad aH \text{ adjacent } bH \text{ iff } a^{-1}b \in HXH.$$

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Theorem

1. *$Cos(\Gamma, H, X)$ is vertex-transitive for all Γ, H, X*
2. *every vertex-transitive graph is $Cos(\Gamma, H, X)$ for some Γ, H, X*

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Theorem (RJ, Širáň)

Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1_G\}$. Further suppose that there are at least $|X| + 1$ distinct ordered pairs $(x, h) \in X \times H$ such that $(xh)^p = 1_G$ for some fixed prime $p > |X| \cdot |H|^2$. Then the coset graph $\Gamma = Cos(G, H, X)$ is a **vertex-transitive non-Cayley graph**.

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No new non-Cayley numbers were discovered by this construction, as the orders of the groups we used contained lots of powers.

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Lemma (RJ, Širáň)

Let $\Gamma = C(G, X)$ *be a localy finite Cayley graph and* p *be a prime. Then the number of closed oriented walks of length* p^n*,* $n \geq 1$*, based at any fixed vertex of* Γ*, is congruent* $\pmod{p}$ *to the number of elements in* X *for which* $x^{p^n} = 1_G$*.*

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Let $\Gamma = C(G,X)$ be a localy finite Cayley graph and p be a prime. Then the number of closed oriented walks of length p^n, $n \geq 1$, based at any fixed vertex of Γ, is congruent $\pmod{p}$ to the number of elements in X for which $x^{p^n} = 1_G$.

Lemma (RJ, Širáň)

Let $\Gamma = C(G,X)$ be a localy finite Cayley graph and p and q be two distinct primes. Let $n = pq$ and let j_n be the number of elements $x \in X$ for which $x^n = 1_G$. Then the number of closed oriented walks of length n, based at any fixed vertex of Γ, is congruent $\pmod{p}$ to $j_n + kq$, where k is a non-negative integer.

Prehistory

Theorem (RJ, Širáň)

Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1_G\}$. Let $p_1^{k_1}, p_2^{k_2}, \ldots, p_r^{k_r}$ be powers of distinct primes, and let ℓ_{p_i}, $1 \leq i \leq r$, denote the number of distinct pairs $(x, h) \in X \times H$ such that $(xh)^{p_i^{k_i}} = 1_G$. Suppose that $\sum_{i=1}^{r} \ell_{p_i} > |X|$, and, for all i, $p_i > \ell_{p_i}|H|$. Then the coset graph $\Gamma = Cos(G, H, X)$ is a **vertex-transitive non-Cayley graph**.

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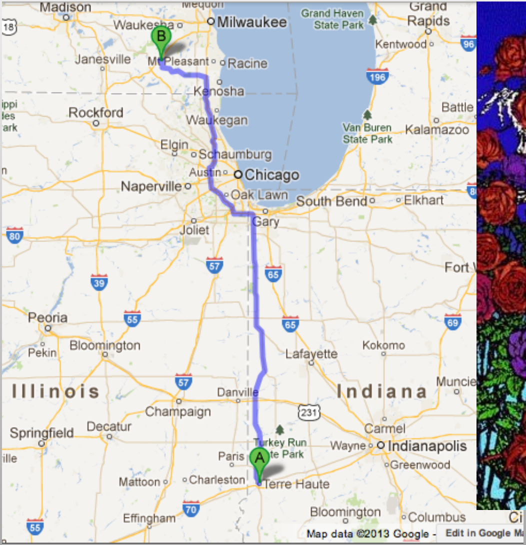
This time we got some new non-Cayley numbers:
19,886, **5,666,226**.

The Dawn of History

Ice Hockey World Champions, May 2002

The Big Journey to the North, August 2002



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Theorem (RJ, Malnič, Marušič)

Let $\Gamma = Cay(G, X)$ be a Cayley graph, and let $b \in V(\Gamma)$. Then the following statements hold.

(i) If $n = p^r$, where p is an odd prime and $r \geq 1$, or $n = 2^r$, where $r \geq 2$, then

$$\mathcal{C}^b_\Gamma(n) \equiv \omega_X(n) \pmod{p}.$$

(ii) If $n = p \cdot q$, where p and q are distinct primes, then

$$\mathcal{C}^b_\Gamma(n) \equiv \omega_X(n) + s \cdot q \pmod{p},$$

where s is a nonnegative integer.

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Let $\Gamma = Cay(G, X)$ be a Cayley graph, $b \in V(\Gamma)$ a base vertex, and p a prime. Then the numbers $\mathcal{W}_\Gamma^b(p)$ and $\mathcal{C}_\Gamma^b(p)$ are congruent modulo p.

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Corollary (RJ, Malnič, Marušič)

Let $\Gamma = Cay(G, X)$ be a Cayley graph, $b \in V(\Gamma)$ a base vertex, and p a prime relatively prime to $|G|$. Then

$$\mathcal{W}_\Gamma^b(p) \equiv \mathcal{C}_\Gamma^b(p) \equiv 0 \pmod{p}.$$

Cycles and Walks in Vertex-Transitive Graphs

Lemma

1. *Let* Γ *be* any *graph, and* n *be a positive integer. Then,*

$$n \mid \sum_{b \in V(\Gamma)} \mathcal{W}_{\Gamma}^{b}(n) \quad \textit{and} \quad n \mid \sum_{b \in V(\Gamma)} \mathcal{C}_{\Gamma}^{b}(n).$$

2. *If* Γ *is vertex-transitive, then* $\mathcal{W}_{\Gamma}^{a}(n) = \mathcal{W}_{\Gamma}^{b}(n)$ *and* $\mathcal{C}_{\Gamma}^{a}(n) = \mathcal{C}_{\Gamma}^{b}(n)$*, for all* $a, b \in V(\Gamma)$*.*

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Corollary

Let Γ *be a vertex-transitive graph, and let* n *be a positive integer relatively prime to* $|V(\Gamma)|$*. Then,*

1. $n \mid \mathcal{W}_\Gamma^b(n)$*, for all* $b \in V(\Gamma)$,
2. $n \mid \mathcal{C}_\Gamma^b(n)$*, for all* $b \in V(\Gamma)$.

Theorem (RJ, Malnič, Marušič)

Let Γ be a connected vertex-transitive graph, $b \in V(\Gamma)$, let $G \leq Aut\ \Gamma$ act transitively on $V(\Gamma)$, m be the order of a vertex-stabilizer in G, and let $X = \{g \in G \mid g(b) \in N(b)\}$. Then the following statements hold.

(i) If p is an odd prime divisor of m and $r \geq 1$ or $p = 2$ divides m and $r \geq 2$, then $\varepsilon_X(p^r) \equiv 0 \pmod{p}$, and for each prime $q \neq p$, there exists a nonnegative integer s such that $\varepsilon_X(pq) + sq \equiv 0 \pmod{p}$.

(ii) If p is an odd prime not dividing m and $r \geq 1$ or $p = 2$ does not divide m and $r \geq 2$, then $\varepsilon_X(p^r) \equiv \mathcal{W}^b_\Gamma(p^r) \pmod{p}$, and, for each prime $q \neq p$, there exists a nonnegative integer s such that $\varepsilon_X(pq) + sq \equiv \mathcal{W}^b_\Gamma(pq) \pmod{p}$.

Walks and Cycles in Petersen

n	5	6	7	8	9	10
$\mathcal{C}^b_{\mathcal{P}}(n)$	12	12	0	24	36	0
$\mathcal{W}^b_{\mathcal{P}}(n)$	12	99	168	759	1764	6315

Applications to Graphs of Given Degree, Diameter, and Girth

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- A **(Δ, D)-graph** is a (finite) graph of *maximum degree* Δ and *diameter* D.
- A **(k, g)-graph** is a (finite) graph of *degree* k and *girth* g.

Moore Bound(s)

$$n(\Delta, D) \leq M(\Delta, D) = \begin{cases} 1 + \Delta\frac{(\Delta-1)^D-1}{\Delta-2}, & \text{if } \Delta > 2 \\ 2D+1, & \text{if } \Delta = 2 \end{cases}$$

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$$n(k, g) \geq M(k, g) = \begin{cases} 1 + k\frac{(k-1)^{(g-1)/2}-1}{k-2}, & g \text{ odd} \\ 2\frac{(k-1)^{g/2}-1}{k-2}, & g \text{ even} \end{cases}$$

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For each odd integer $k \geq 3$ there is an infinite sequence of values of g such that the excess e of any vertex-transitive graph with valency k and girth g satisfies $e > g/k$.

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Theorem (Exoo, RJ, Mačaj, Širáň)

For any fixed pair of degree Δ and defect δ, the set of diameters D for which there might exist a vertex-transitive (Δ, D)-graph of defect not exceeding the defect δ is of measure 0 with respect to the set of all positive integers.

Cayley (k, g)-Graphs

Theorem (Exoo, RJ, Širáň)

For every $k \geq 2, g \geq 3$, *there exists a* **Cayley** (k, g)*-graph.*

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- constructing Cayley graphs from 1- and 2-factorizations of (k, g)-graphs

Graphs to Groups

Theorem (Exoo, RJ, Širáň)

Let G be a k-regular graph of girth g whose edge set can be partitioned into a family $\mathcal{F}$ of k_1 1-factors, F_i, $1 \leq i \leq k_1$, and k_2 oriented 2-factors F_i, $k_1+1 \leq i \leq k_1+k_2$ (where $k_1+2k_2=k$). If $\Gamma_{\mathcal{F}}$ is the finite permutation group acting on the set $V(G)$ generated by the set

$$X = \{\delta_{F_i} \mid 1 \leq i \leq k_1\} \cup \{\sigma_{F_i} \mid k_1+1 \leq i \leq k_1+k_2\} \cup \{\sigma_{F_i}^{-1} \mid k_1+1 \leq i \leq k_1+k_2\},$$

then the Cayley graph $Cay(\Gamma_{\mathcal{F}}, X)$ is k-regular of girth at least g.

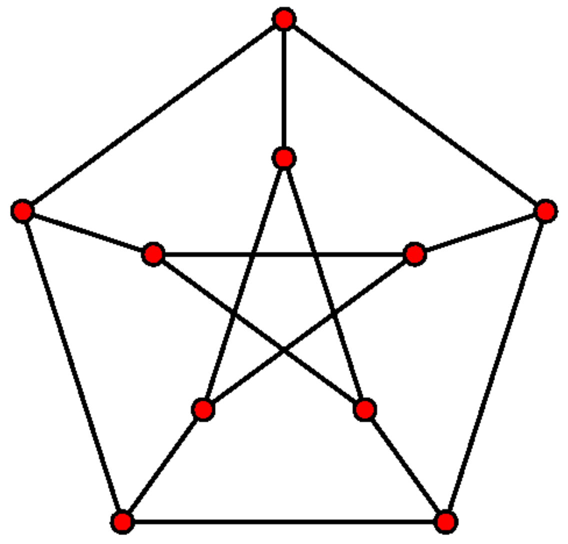

Figure: Smallest $(3,5)$-graph

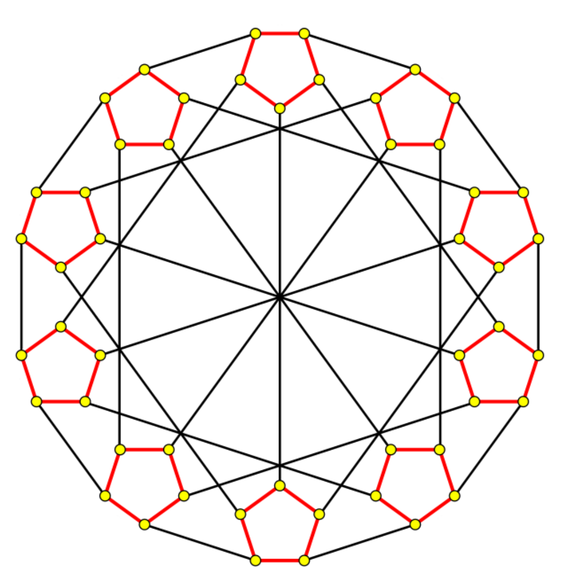

Figure: Smallest Cayley $(3, 5)$-graph

Decompositions of Vertex-Transitive Graphs into Cycles

Observation 1.: Every Cayley graph $C(G, X)$ decomposes into k_1 1-factors and k_2 2-factors (where the cycles in each of the 2-factors are of the same length), where k_1 is the number of involutions in X and k_2 is the number of non-involutions in X, $k_1 + k_2 = |X|$.

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A k-regular graph Γ is Cayley if and only if there exists a partition of $E(\Gamma)$ into 1- and 2-factors consisting of cycles of equal length such that the corresponding vertex-transitive graph is of order equal to the order of Γ.

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Observation 2. The edges of these 1- and 2-factors are orbits of right-multiplication automorphisms of the underlying Cayley graph; $\varphi_x(a) = ax$, $x \in X$.

Marušič:
Every vertex-transitive finite graph has a regular automorphism.

The Polycirculant Conjecture

Marušič:

Every vertex-transitive finite graph has a regular automorphism.

My List of Naive Questions

- Is the conjecture true for every vertex-transitive automorphism group of a vertex-transitive graph?
- Is the conjecture true if we put a limit on the order of a vertex-stabilizer in a vertex-transitive graph?
- Is the conjecture true for quasi-Cayley graphs?

  

Happy

$(2^2 \cdot 3 \cdot 5)$ - Birthday!!!!

Dragan