Half-arc-transitive and Semi-symmetric Graphs

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In honour of Dragan Marušič

A brief history (of times)

- Late 1970s – Dragan and Marston lived in UK cities just 50km apart, doing their doctorates, but didn't meet
- 1996 – joint paper (with Brian Alspach and Xu Mingyao), on 2-arc-transitive circulants, but still didn't meet
- 1998 – first met, at SIGMAC meeting (Flagstaff, Ariziona)
- 1999 – Marston's first visit to Slovenia (Bled conference)
- 2000 – Dragan's first visit to Auckland
- 2001 – Marston's first visit to Ljubljana [Note later]
- 2011 – Conference at Fields Institute [Note later]

A tribute

I pay tribute to Dragan for his being a number of things:

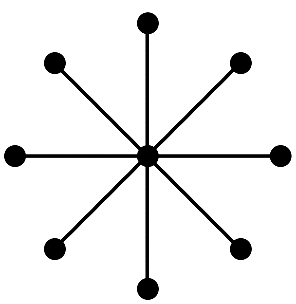
- An excellent mathematician
- A dedicated family man
- A wonderful colleague and friend to many of us
- Keen on health and personal fitness/sports
- A leader of discrete mathematics in this part of Europe
- A strong supporter of Koper/Primorska
- A modest and mature individual ...

Summer School, Rogla, June 2011:

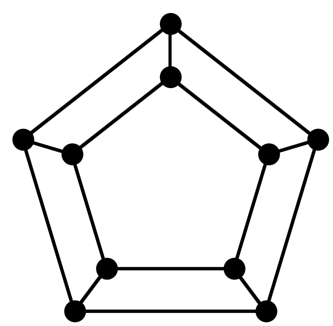
Symmetries of graphs

A symmetry (or *automorphism*) of a connected simple graph X is any permutation of its vertices preserving adjacency. Under composition, these symmetries form the *automorphism group* of X, denoted by $\mathrm{Aut}\, X$.

If $\mathrm{Aut}\, X$ is transitive (i.e. has a single orbit) on vertices, then X is *vertex-transitive.* Similarly, if $\mathrm{Aut}\, X$ is transitive on edges, then X is *edge-transitive.*



is ET not VT



is VT not ET

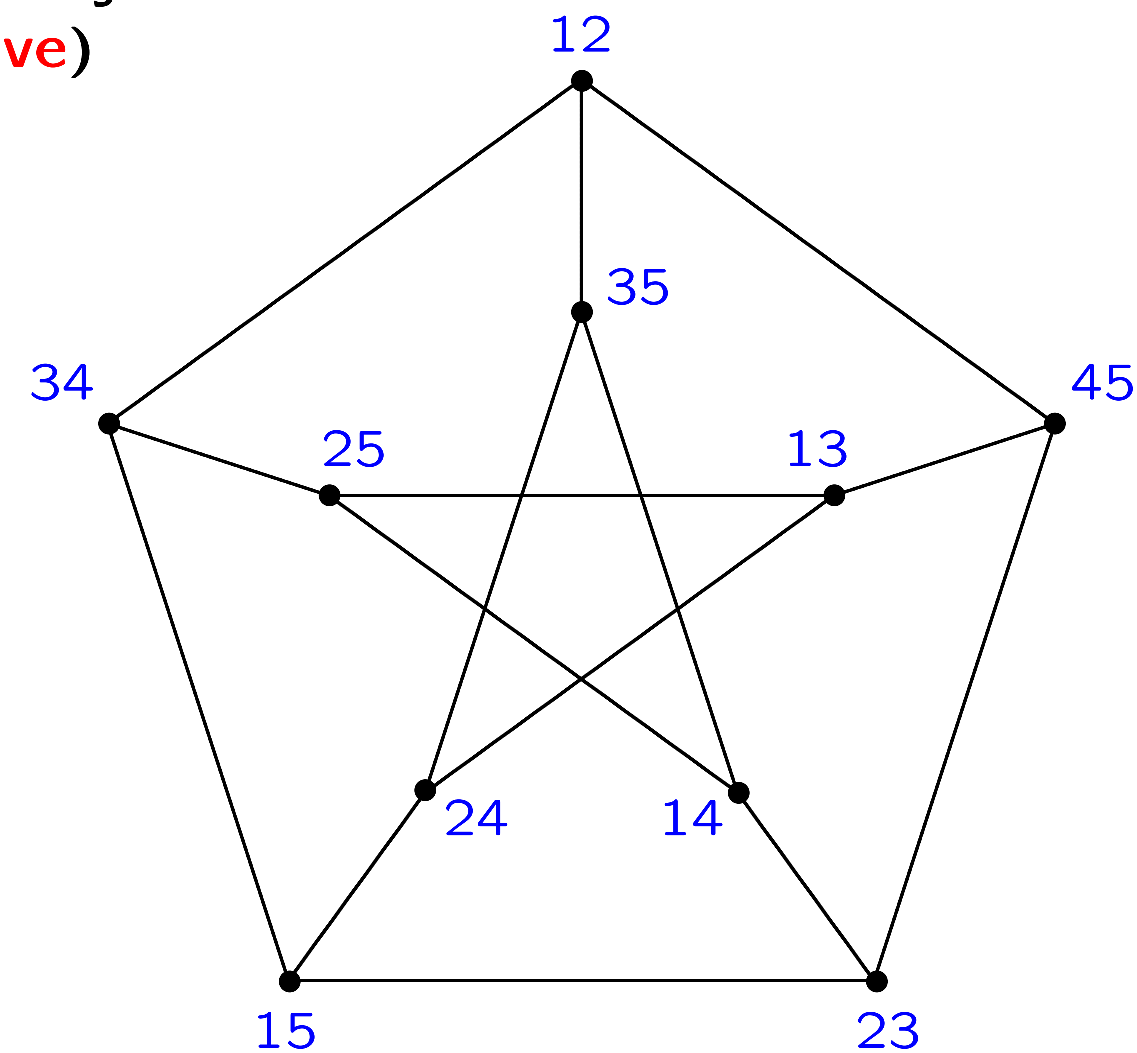
Similarly, if $\operatorname{Aut} X$ is transitive on ordered pairs of adjacent vertices, then X is *arc-transitive*, or *symmetric*.

More generally, if $\operatorname{Aut} X$ is transitive on s-arcs (directed walks of length s in X in which any two successive vertices are adjacent and any three successive vertices are distinct), then X is *s-arc-transitive*.

Examples

- C_n is vertex-transitive, and s-arc-transitive for all s
- K_n is vertex-, arc- and 2-arc- but not 3-arc-transitive
- $K_{n,n}$ is 3-arc- but not 4-arc-transitive

The Petersen graph is symmetric
(in fact 3-arc-transitive)



Every 3-arc has form

ab — cd — ae — bc

Two ‘half-way houses’

A graph X is semi-symmetric if it is edge-transitive but not vertex-transitive. For example, the ‘star’ graph earlier is semi-symmetric (but with one vertex of valence/degree different from the others). The smallest 3-valent example is the Gray graph (with 54 vertices).

A graph X is half-arc-transitive if it is vertex-transitive and edge-transitive but not arc-transitive. The smallest example is the Holt graph (which is 4-valent on 27 vertices).

Constructions

There are various ways of constructing graphs that are symmetric, or semi-symmetric, or half-arc-transitive. Some are ad hoc, while some are systematic.

For example, every connected symmetric graph X can be constructed by taking a group G, a subgroup H and an element $a \in G$ such that $a^2 \in H$ and $G = \langle H, a \rangle$: take the vertices of X as right cosets Hg (for $g \in G$), and join Hx to Hy by an edge whenever $xy^{-1} \in HaH = \{h_1ah_2 : h_1, h_2 \in H\}$. This is the (Sabidussi) double coset graph construction.

The group G induces automorphisms of X by multiplication, with H being the stabiliser of the vertex (labelled) H.

Example: 3-valent symmetric graphs

Infinitely many 3-valent symmetric graphs are constructible from groups G that can be generated by two elements a and h with $a^2 = h^3 = 1$ (i.e. quotients of the modular group $C_2 * C_3 \cong \mathrm{PSL}(2, \mathbb{Z})$): just take $H = \langle h \rangle \cong C_3$.

The element a interchanges the vertex H with its neighbour Ha, while the element h induces a cyclic rotation of the three neighbours Ha, Hah and Hah^2 about the vertex H.

This gives all such graphs that admit a subgroup of automorphisms acting regularly on the arcs (arc-regular graphs).

Example: 3-valent **semi**-symmetric graphs

Similarly, 3-valent semi-symmetric graphs are constructible from groups G that can be generated by two elements x and y with $x^3 = y^3 = 1$ (i.e. quotients of $C_3 * C_3$).

Take $H = \langle x \rangle \cong C_3$ and $K = \langle y \rangle \cong C_3$, and let the edges be all pairs of the form $Hu — Kv$ for $u, v \in G$ with $Hu \cap Kv \neq \emptyset$ (or equivalently, all pairs of the form $Hg — Kx^i g$ for $g \in G$).

All finite connected 3-valent semi-symmetric graphs can be constructed from one of Goldschmidt's 15 finite primitive amalgams of index $(3, 3)$, of which $C_3 * C_3$ is the 'smallest'.

Some more history

- Late 1990s – MC and PhD student Peter Dobcsányi used computer techniques to find all 3-valent symmetric graphs on up to 768 vertices (extending/correcting Foster census)

- 2001 (Ljubljana) – Primož Potočnik & MC began a similar classification of 3-valent semi-symmetric graphs, which immediately resulted in the discovery of the Ljubljana graph (the 3rd smallest example, on 112 vertices)

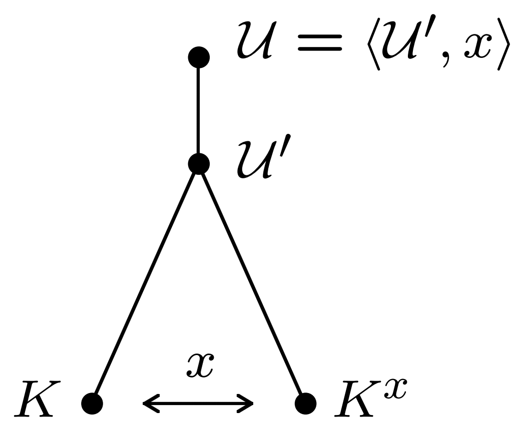
- Sandi Malnič and Dragan Marušič joined MC & PP to help complete the classification of 3-valent semi-symmetric graphs on up to 768 vertices, including several previously undiscovered examples, & analysis of quotients and covers [see paper in *J. Algebraic Combinatorics* 23 (2006)]

A birthday present for Dragan

Announcement: Complete determination of all 3-valent semi-symmetric graphs on up to $10,000$ vertices.

This achieved by Marston and Primož P, using the recently improved `LowIndexNormalSubgroups` algorithm to find all small smooth quotients of the 15 Goldschmidt amalgams (plus a bit of trickery for the largest amalgams).

Key approach: In this context (and others) we have a universal group $\mathcal{U}$ for arc-transitive actions on graphs of a particular kind (e.g. the modular group $C_2 * C_3$ for one-arc-regular cubic graphs). In such cases, $\mathcal{U}$ has a subgroup $\mathcal{U}'$ of index 2 which is a universal group for the 'half-way' action (e.g. semi-symmetric or half-arc-transitive). And to find examples, we seek quotients $\mathcal{U}'/K$ by subgroups K that are normal in $\mathcal{U}'$ but not normal in $\mathcal{U}$:



Question: What about half-arc-transitive graphs?

Recall: half-arc-transitive graphs are VT and ET but not AT
[In particular, the valency must be even]

These are harder to construct, but still possible. We can specify the valency, and even the stabilizer of a vertex under the action of some (sub)group of automorphisms that acts transitively on vertices and edges but not on arcs.

Until 1999, all known examples had abelian vertex-stabiliser.
Then Dragan asked me the obvious question, at Bled ...

First known half-arc-transitive graph with non-abelian vertex-stabiliser

Discovered by Dragan & MC at the 4th Slovenian International Conference on Graph Theory, at Lake Bled, in 1999.

The graph is 4-valent on 10,752 vertices. Its automorphism group is a group order 86016, generated by two elements a and b of orders 8 and 24, with vertex-stabiliser $H = \langle a^{-1}b, a^{-2}ba, a^{-3}ba^2 \rangle \cong D_4$ (dihedral of order 8).

Technical point for the experts: $\{Ha, Hb\}$ is a non-self-paired sub-orbit of the action of G on the coset space $(G : H)$.

More questions about $\frac{1}{2}$-arc transitive graphs

Dragan raised these questions at a workshop at the Fields Institute (Toronto) in October 2011:

- Are there other examples?
- Are there other examples with larger non-dihedral vertex-stabiliser?

Surprisingly, it turns out there's a second 4-valent example on 10,752 vertices (with vertex-stabiliser D_4), not isomorphic to the first.

Also there's one on 21,870 vertices, with similar properties.

Another birthday present for Dragan

Announcement: A new half-arc-transitive 4-valent graph, on $90 \cdot 3^{10}$ vertices, with vertex-stabiliser $D_4 \times C_2$ (of order 16).

Graph found by Primož P, as a 3^{10}-fold cover of a symmetric 4-valent graph on 90 vertices, and then proved by MC to be half-arc-transitive (using local analysis of the graph and study of words in the universal group action).

Apologies for not being able to draw it ... **:-)**