Semiregular automorphisms of arc-transitive graphs

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A few definitions

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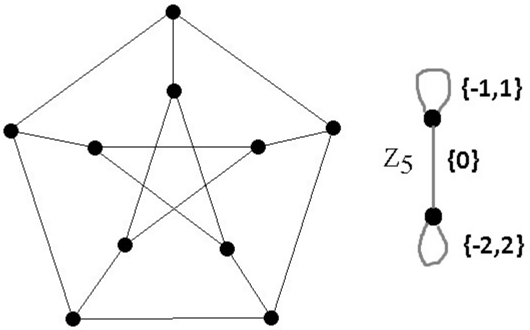
Equivalently, g is semiregular if all of its cycles have the same length.

Semiregular automorphisms



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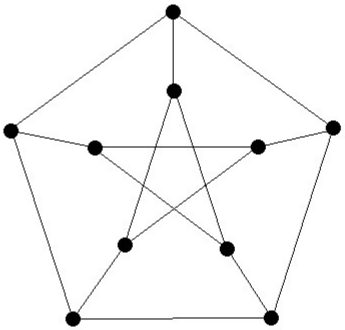

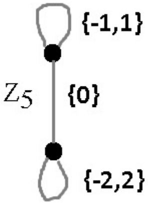

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The larger the semiregular group of automorphism, the more compact the definition. In the “best” case (a regular group) we have a Cayley graph.

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Known for graphs of order p^k and $2p^2$ (Marušič, Scapellato 1981,1998).

Known for graphs of square-free order. (Dobson, Malnič, Marušič, Nowitz 2007).

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2-*arc-transitive graphs and arc-transitive graphs of prime valency admit non-trivial semiregular automorphisms.*

Recent progress

Theorem (Giudici, V. 2013)

An arc-transitive graph of valency twice a prime admits a non-trivial semiregular automorphism.

Sketch of the proof

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Since N is not semiregular, $q = p$ or $q = 2$. In the first case, we use the following result.

Theorem (Praeger, Xu 1989)

Let p be a prime and let Γ be a $2p$-valent G-arc-transitive graph such that G has an abelian normal p-subgroup which is not semiregular on the vertices of Γ. Then $\Gamma \cong \mathrm{C}(p, r, s)$ for some $r \geq 3$ and $1 \leq s \leq r - 1$.

Sketch of the proof, II

We can assume that N is a 2 group and Γ/N is p-valent. We use the following result.

Theorem (Burness, Giudici 2013)

Let p be a prime and let Γ be a G-arc-transitive graph of valency p. Then one of the following occurs.

1. *G has a semiregular element of odd order,*
2. *$|\mathrm{V}(\Gamma)|$ is a power of 2,*
3. *$p = 11$, G contains a semiregular normal 2-subgroup M (possibly $M = 1$), such that $G/M \cong M_{11}$, $G_x \cong \mathrm{PSL}(2, 11)$ and $\Gamma/M \cong K_{12}$. In particular Γ is G-2-arc-transitive.*

What we actually proved

Corollary (Giudici, V. 2013)

Let Γ *be a* G*-arc-transitive graph of valency* $2p$ *such that* G *has an abelian minimal normal subgroup. Then one of the following occurs:*

1. G *has a semiregular element,*
2. $\Gamma \cong \mathrm{C}(p, r, s)$ *for some* $r \geq 3$ *and* $1 \leq s \leq r - 1$*,*
3. $p = 11$ *and* Γ *is unworthy.*

Together with Jing Xu's result, this ends the proof.

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Corollary (Giudici, V. 2013)

Let Γ be a G-arc-transitive graph of valency $2p$ such that G has an abelian minimal normal subgroup. Then one of the following occurs:

1. *G has a semiregular element,*
2. *$\Gamma \cong \mathrm{C}(p, r, s)$ for some $r \geq 3$ and $1 \leq s \leq r-1$,*
3. *$p = 11$ and Γ is unworthy.*

Together with Jing Xu's result, this ends the proof.

It would be interesting to go through Xu's proof to see when she actually needs $\mathrm{Aut}(\Gamma)$ rather than G.

Further work

Theorem (V. 2013)

An arc-transitive graph of valency 8 *admits a non-trivial semiregular automorphism.*

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For valency pq, the bottleneck seems to be solvable groups.

Happy birthday Dragan!