UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

Zaključna naloga
(Final project paper)

Raznoterosti, Grassmannian raznoterosti in ireducibilnost

Varieties, Grassmannian Varieties and Irreducibility

Ime in priimek: Ratko Darda
Študijski program: Matematika
Mentor: Prof. Bojan Kuzma, PhD

Koper, junij 2015

Ključna dokumentacijska informacija

Ime in PRIIMEK: Ratko DARDA

Naslov zaključne naloge: Raznoterosti, Grassmannian raznoterosti in ireducibilnost

Kraj: Koper

Leto: 2015

Število listov: 34 Število referenc: 7

Mentor: Prof. Bojan Kuzma, PhD

Ključne besede: raznoterosti, regularne preslikave, Grassmannian raznoterosti, ireducibilne raznoterosti

Math. Subj. Class. (2010): 14A10

Izvleček:
V zaključni nalogi predstavimo afine in projektivne raznoterosti. Raznoterosti so definirane kot množice skupnih ničel neke neprazne družine polinomov. Obstajajo afine in projektivne raznoterosti. Opremimo jih lahko s topologijo Zariskega, v kateri so zaprte množice natanko vse podraznoterosti. V nalogi so definirane preslikave med raznoterostimi. Razcepne raznoterosti so raznoterosti ki so unija pravih podraznoterosti. Vsako raznoterost lahko razcepimo na končno unija nerazcepnih raznoterosti. Posebej je obravnavana skupina Grassmannskih raznoterosti, ki so definirane kot množice množice množice vseh d-razsežnih podprostorov n-razsežnega vektorskega prostora. Pokazano je da so Grassmannske raznoterosti nerazcepne.

Key words documentation

Name and SURNAME: Ratko DARDA

Title of final project paper: Varieties, Grassmannian Varieties and Irreducibility

Place: Koper

Year: 2015

Number of pages: 34 Number of references: 7

Mentor: Prof. Bojan Kuzma, PhD

Keywords: varieties, Grassmannian varieties, regular maps, irreducibility

Math. Subj. Class. (2010): 14A10

Abstract:
In the final project paper we studied affine and projective varieties. Varieties are defined as the vanishing sets of collections of polynomials. We distinguish between affine and projective varieties. We can endow them with so called Zariski topology, where subvarieties are closed sets. In the project we defined and studied maps between varieties. Varieties that are not union of proper subvarieties are called irreducible, otherwise they are reducible. Every variety is finite union of irreducible varieties. In the project we studied Grassmannian varieties, which are sets of d-dimensional subspaces of n-dimensional vector space. It is shown that Grassmannian varieties are irreducible.

Acknowledgement

Thanks to professor Bojan Kuzma for his useful advices and help.
Thanks to the personnel of FAMNIT for all the help they gave me.
Thanks to my family, relatives, friends and everyone else who has supported me during all these years.
My studies would be impossible without the scholarship I have been receiving and I would like to thank FAMNIT for that.

Contents

List of Abbreviations

i.e. that is
e.g. for example
RHS right hand side
LHS left hand side

1 Introduction

In the final project we will present some classical notions of algebraic geometry. Varieties are classically defined as the vanishing sets of polynomials. Let us give a short overview what is going to be presented in the final project. In Chapter 2 we recall some notions and theorems we will use in the rest of work.
In Chapter 3 we explain different types of varieties, natural topology on them and give some examples.
In Chapter 4 and Irreducibility we explain what does it mean for varieties to be isomorphic, and we explain what are atomic varieties.
In Chapter 5 we explain an important class of varieties, called Grasmmannian varieties and prove their irreducibility. Our work mostly follows [1].

2 Preliminaries

2.0.1 Notation

By $\mathbb{N}$ we denote the set of positive integers.
By $\mathbb{R}$ we denote the set of real numbers.
If A_i are pairwise disjoint, we sometimes write $\coprod A_i$ for $\bigcup A_i$.

2.0.2 Set Theory

Definition 2.1. Relation $\leq$ on a set X is said to be:

1. reflexive, if $\forall a \in X$ we have $a \leq a$;
2. antisymmetric, if $\forall a, b \in X$ we have $a \leq b \wedge b \leq a \implies a = b$;
3. transitive, if $\forall a, b, c \in X$ such that $a \leq b, b \leq c$ we have $a \leq c$;
4. total, if $\forall a, b \in X$ we have that some of $a \leq b$ and $b \leq a$ is true.

Definition 2.2. Partially ordered set is a set X together with reflexive, antisymmetric and transitive relation $\leq$.

Definition 2.3. Totally ordered set is a set X together with a total and transitive relation $\leq$.

Lemma 2.4 (Zorn's lemma). *Let X be a partially ordered set such that each chain (i.e. totally ordered subset) has an upper bound. Then X has at least one maximal element.*

Proof. See [7], page 63. □

2.0.3 Field Theory

Definition 2.5. Let k be a field. We say that k is algebraically closed if every non-constant polynomial with coefficients in k has a zero in k.

Remark 2.6. An important theorem, which gives some motivation for studying only algebraically closed fields in algebraic geometry, is that for any field there exists extension of it that is algebraically closed. For a proof, see [4], page 231, theorem 2.5.

Here we state simple observation about algebraically closed fields:

Proposition 2.7. *If k is algebraically closed then k is an infinite field.*

Proof. Suppose k was finite. Let $a_1, \dots, a_n$ be all elements of k. Then polynomial

$$(x - a_1)(x - a_2) \cdot \dots \cdot (x - a_n) + 1$$

has no roots in k, a contradiction. □

2.0.4 Commutative Algebra

Definition 2.8. A commutative ring with unity $(A, +, \cdot, 1)$ is a set A equipped with two bilinear operations $+$ and $\cdot$ such that $(A, +)$ is an abelian group, $(A, \cdot, 1)$ is a commutative monoid with 1 as the unity element and for any three elements a, b, c from A we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

We often drop the multiplication sign. We use term ring for commutative ring with unity and write A for commutative ring $(A, +, \cdot, 1)$.

Definition 2.9. Let A and B be rings. A ring homomorphism $f : A \to B$ is a map such that $f(1_A) = 1_B$ and for every $x, y \in A$ we have $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, where 1_A and 1_B are unity elements in A and B, respectively.

Definition 2.10. An ideal $\mathfrak{a}$ of ring A is its subset such that $(\mathfrak{a}, +)$ is a subgroup of $(A, +)$ and for every $a \in \mathfrak{a}$ and $x \in A$, the product ax is in $\mathfrak{a}$. An ideal is maximal if it is proper subset of A and if it is maximal with respect to inclusion. An ideal $\mathfrak{a}$ is prime if whenever $ab \in \mathfrak{a}$ for $a, b \in A$, then at least one of a, b is in $\mathfrak{a}$. The smallest ideal containing all ideals $\mathfrak{a}_i, i \in I$ is denoted by $\sum \mathfrak{a}_i$. We can see that:

$$\sum_{i \in I} \mathfrak{a}_i = \{a_{i_1} + \dots + a_{i_n} | n \in \mathbb{N}, \{i_1, \dots, i_n\} \subset I, a_{i_j} \in \mathfrak{a}_{i_j}\}$$

Definition 2.11. An integral domain is a ring A such that for any of its elements a, b with $ab = 0$ we must have $a = 0$ or $b = 0$.

Proposition 2.12. *Let A be a ring, B an integral domain, and $f : A \to B$ be a ring homomorphism. Then*

$$\ker f = \{x \in A | f(x) = 0\}$$

is a prime ideal in A.

Proof. Assume $xy \in \ker f$. Then $f(x)f(y) = 0$ implies at least one of $f(x)$ or $f(y)$ is zero. Therefore at least one of x or y is in $\ker f$. □

Proposition 2.13. *Given proper ideal $\mathfrak{a}$ of A, there exists a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$.*

Proof. (from [3])
The set of all proper ideals containing $\mathfrak{a}$ is non-empty and it satisfies the condition for the Zorn's lemma. Namely, if we have a chain of proper ideals $I_1 \subset I_2 \subset \cdots$ containing $\mathfrak{a}$, their union $I = \bigcup_{i=1}^{\infty} I_i$ contains $\mathfrak{a}$ and it is a proper ideal itself: for any two elements $x, y \in I$, there exists j such that $x, y \in I_j$, so their sum $x + y \in I_j \subset I$, and if $a \in A$ and $x \in I$, we can find j such that $x \in I_j$, so $ax \in I_j \in I$.
By Zorn's lemma, there exists a maximal element $\mathfrak{m}$ in the collection. This element surely must be a maximal ideal in A. □

Corollary 2.14. *Let a be a non-unit element in ring A. Then there exists maximal ideal which contains a.*

Proof. a non-unit $\iff$ (a) is a proper ideal in A. Apply 2.13 on (a). □

By $A[x_1, \ldots, x_n]$ we denote the ring of polynomials in n variables with coefficients in A.

Definition 2.15. Let $\mathfrak{a}$ be an ideal in ring A. We define the radical of the ideal

$$\mathrm{rad}(\mathfrak{a}) = \{r \in A | \exists n \in \mathbb{N}, r^n \in \mathfrak{a}\}$$

For an ideal $\mathfrak{b} \in A[x_1, \ldots, x_n]$ we denote by $V(S)$ the set of common zeros of all polynomials from $\mathfrak{b}$ in A^n (where we calculate the value of polynomial at a vector simply by replacing x_i for vector's i-th coordinate and sum and multiply everything inside A).
For a set $S \subset A^n$ we denote by $I(S)$ the set of all polynomials in $A[x_1, \ldots, x_n]$ vanishing on the whole of S. Obviously, $I(S)$ is an ideal for every subset S.

Theorem 2.16 (Hilbert Nullstellensatz). *Let k be a field and $\mathfrak{a}$ be an ideal in $k[x_1, x_2, \ldots, x_n]$. If k is algebraically closed then:*

$$I(V(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a}).$$

Proof. See [1], page 49, theorem 5.1. □

Remark 2.17. If k is not algebraically closed, then statement fails to be true. If $k = \mathbb{R}$, $n = 1$ and $\mathfrak{a} = (x^2 + 1)$. We see that $\mathrm{rad}(\mathfrak{a}) = \mathfrak{a}$, because $x^2 + 1$ is irreducible. But $V(x^2 + 1) = \emptyset$, $I(\emptyset) = \mathbb{R}[x] \neq \mathrm{rad}(\mathfrak{a})$.

Definition 2.18. Ring A is called Noetherian, if it satisfies ascending chain condition on ideals, that is: given sequence of ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ there exists integer m such that $I_m = I_{m+1} = \cdots$.

Proposition 2.19. *A is Noetherian ring* $\iff$ *every ideal of A is finitely generated.*

Proof. Suppose A is Noetherian. Suppose there exists ideal I of A which is not finitely generated. Take an element x_1 of I, and consider the ideal (x_1). Since I is not finitely generated $\exists x_2 \in I \setminus (x_1)$. Then $(x_1) \subsetneq (x_1, x_2) \subsetneq I$ since I is not finitely generated. Continuing process we see that there exists an infinite non-terminating chain of ideals. Conversely, let every ideal be finitely generated and consider an infinite chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$. Their union $I = \bigcup_{i=1}^{\infty} I_i$ is also an ideal, as we have seen in the proof of 2.13, which, by assumption, is finitely generated. Take a set of generators: $\{x_1, x_2, \ldots, x_n\}$ for I. Since all x_i are in the union, there must be some indices j_i such that $x_i \in I_{j_i}$. But that means that all x_i are in I_j, where $j = \max(j_1, j_2, \ldots, j_n)$. Therefore the sequence terminates at I_j. □

Theorem 2.20 (Hilbert's Basis Theorem). *Let A be a Noetherian ring. Then* $A[x_1, x_2, \ldots, x_n]$ *is Noetherian as well.*

Proof. (taken from [3])
Let us prove that if A is Noetherian that $A[x]$ is Noetherian as well. By induction this will imply that $A[x_1, x_2, \ldots, x_n]$ is Noetherian.
Suppose there exists ideal $\mathfrak{a}$ of $A[x]$ which is not finitely generated. Choose a sequence $f_1, f_2, \ldots, \subset \mathfrak{a}$ in the following way: f_1 is a polynomial of the least degree in $\mathfrak{a}$, and f_{i+1} is a polynomial of the least degree not contained in $(f_1, f_2, \ldots, f_i)$. Our sequence is infinite, because $\mathfrak{a}$ is not finitely generated.
Let a_j be the leading coefficient in f_j. A is a Noetherian ring, therefore $\mathfrak{b} = (a_1, a_2, \ldots)$ is finitely generated. Let $(a_1, a_2, \ldots, a_m)$ generate $\mathfrak{b}$. We claim $\mathfrak{a} = (f_1, f_2, \ldots, f_m)$, which will bring us a contradiction.
We can write $a_{m+1} = \sum_{j=1}^{m} u_j a_j$ for some $u_j \in A$. Define

$$g = \sum_{j=1}^{m} u_j f_j x^{\deg f_{m+1} - \deg f_j} \in (f_1, \ldots, f_m)$$

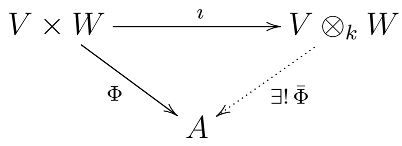
and notice that it has the same leading monomial as f_{m+1}, so the difference $h = f_{m+1} - g \in \mathfrak{a}$, but not in $(f_1, f_2, \ldots, f_m)$ and $\deg h < \deg g$, a contradiction. □

2.0.5 Tensor Product

We recall following facts about tensor products of vector spaces.

Definition 2.21. Let V, W be finite-dimensional vector spaces over a field k. Then, their tensor product, $V \otimes_k W$ is a k-vector space together with bilinear map $\imath \colon V \times W \to$

$V \otimes_k W$ such that for each k-vector space A and each bilinear $\Phi : V \times W \to A$ there exists a unique bilinear map $\bar{\Phi} : V \otimes_k W \to A$ such that the diagram



commutes.

Lemma 2.22. *For finite dimensional vector spaces V, W there exists their tensor product. Moreover,* $\dim_k V \otimes W = (\dim V) \cdot (\dim W)$.

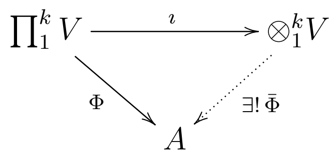
Proof. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be basis in V and W, respectively. Choose the nm pairs (v_i, w_j) as a basis for k-linear space U and denote it shortly as $v_i \otimes w_j := (v_i, w_j)$. Define a map $\imath\colon V \times W \to U$ first on basis vectors by

$$\imath(v_i, w_j) := v_i \otimes w_j$$

and then extend it bilinearly. Thus, $\imath(\sum \alpha_i v_i, \sum_j \beta_j w_j) = \sum_{ij} \alpha_i \beta_j v_i \otimes w_j$.

Hence, if $\Phi\colon V \times W \to A$ is a bilinear map, then $\Phi(\sum_i \alpha_i v_i, \sum_j \beta_j w_j) = \sum \alpha_i \beta_j \Psi(v_i, w_j)$, so we may uniquely define linear $\bar{\Psi}\colon V \otimes_k W \to A$ on basis vectors by $\bar{\Psi}\colon v_i \otimes w_j \mapsto \Psi(v_i, w_j)$ to achieve that the diagram from the definition commutes. $\square$

Given a vector space V over a field k, we recursively define $\otimes_1^d V$ by $\otimes_1^1 V := V$ and $\otimes_1^{d+1} V := (\otimes_1^d V) \otimes V$. We remark that $\imath\colon \prod_1^d V \to \otimes_1^k V$, defined by $\imath\colon (v_1, \ldots, v_d) \mapsto v_1 \otimes \cdots \otimes v_d$ is a multilinear map, and has the following characterizing property: if $\Psi : \prod_1^k V \to A$ is a multiliear map to a k-vector space A then there exists a unique linear $\bar{\Phi} : \otimes_1^k V \to A$ so that the diagram



commutes.

2.0.6 Wedge product

We define

$$\wedge^d V = \underbrace{V \otimes \cdots \otimes V}_{d} / Z$$

where Z is the subspace generated by products $x_1 \otimes \cdots \otimes x_d$ and some of two x_i-s are the same. We denote the image of $u_1 \otimes \cdots \otimes u_d$ under the quotient map by $u_1 \wedge \cdots \wedge u_d$. We call it wedge product of vectors $u_1, \ldots, u_d$ (respecting order).

Lemma 2.23. *Let $u_1, \dots, u_d$ be a vectors from V. Then*

$$u_{\pi(1)} \wedge \cdots \wedge u_{\pi(d)} = \sigma(\pi) u_1 \wedge \cdots \wedge u_d,$$

where $\sigma(\pi))$ is the sign of permutation π.

Proof. Let's see this is true if π is a transposition. Take two wedge products such that second is obtained by replacing i-th and j-th coordinate in the first one: $u = x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \cdots \wedge x_d$ and $v = x_1 \wedge \cdots \wedge x_j \cdots \wedge x_i \wedge \cdots \wedge x_d$. Let u' be a vector which is obtained from u by replacing x_i with x_j in the wedge product. Similarly define v', here we replace x_j by x_i in the second wedge product. We have $u', v' = 0$. Therefore:

$$u + v = u + u' + v + v' = (x_1 \wedge \cdots \wedge \underbrace{(x_i + x_j)}_{i-\text{th position}} \wedge \cdots \wedge x_j \wedge \cdots \wedge x_d)$$

$$+(x_1 \wedge \cdots \wedge \underbrace{(x_i + x_j)}_{i-\text{th position}} \wedge \cdots \wedge x_i \wedge \cdots \wedge x_d) = 0.$$

We get $u + v = 0$ as claimed.
Any permutation is product of transpositions, so the statement follows. □

Proposition 2.24. *Let $v_1, \dots, v_n$ be a basis for V. Vectors $v_{i_1} \wedge \cdots \wedge v_{i_d}$, where $1 \leq i_1 < \cdots < i_d \leq n$ form a basis of the vector space $\wedge^d V$.*

Proof. Let $f_1, \dots, f_n$ be the corresponding dual basis in V^*. Then, the map

$$\phi : V \times V \times \cdots \times V \to k$$

$$(v_1, \dots, v_d) \mapsto \det([f_i(v_j)]_{1 \leq i,j \leq d})$$

is alternate and multilinear, and annihilates all wedge products $v_{i_1} \wedge \cdots \wedge v_{i_d}$ except when $(i_1, \dots, i_d) = (1, \dots, d)$.

Thus, this map induces a well defined linear functional on wedge product, which annihilates all basis vectors in wedge product, except for $v_1 \wedge \cdots \wedge v_d$.

Likewise for other basis wedge vectors. This shows that $v_{i_1} \wedge \cdots \wedge v_{i_d}$ are truly linearly independent. □

Corollary 2.25. *Dimension of $\wedge^d V$ is $\binom{n}{d}$.*

Proof. Follows from the previous proposition. □

Proposition 2.26. *A vector $w \in \wedge^d V$ is divisible by $v \in V$, i.e. $w = v \wedge \phi$ for some $\phi \in \wedge^{d-1} V$, iff $w \wedge v = 0 \in \wedge^{d+1} V$.*

Proof. If v divides w then $w \wedge v = 0$.
Assume $w \wedge v = 0$. Write $w = \sum a_{j_1,\dots,j_d} v_{j_1} \wedge \cdots \wedge v_{j_d}$ and $v = \sum_1^s b_i v_i$ as the sum of basis vectors where we remove those b_i equal to zero and without loss of generality suppose that we take vectors with the least indices. We have

$$w \wedge v = \sum a_{j_1,\dots,j_d} v_{j_1} \wedge \cdots \wedge v_{j_d} \wedge \sum b_i v_i =$$

$$\sum b_i a_{j_1,\dots,j_d} v_{j_1} \wedge \cdots \wedge v_{j_d} \wedge v_i$$

Now whenever $a_{j_1,\dots,j_d}$ is nonzero we see that each v_i must appear among $\{v_{j_1}, \dots, v_{j_d}\}$ therefore $w = v_1 \wedge \cdots \wedge v_s \wedge \phi'$. But this means $w = b_1 v_1 \wedge \cdots \wedge b_s v_s \wedge \frac{1}{\prod_{i=1}^s b_i} \phi'$. So for ϕ we take $\frac{1}{\prod_{i=1}^s b_i} \phi'$. □

Lemma 2.27. *Let W be a d-dimensional subspace of V. Let $u_1, \dots, u_d$ and $v_1, \dots, v_d$ be its two basis, and A the change matrix from u to v. Then*

$$u_1 \wedge \cdots \wedge u_d = \det(A) v_1 \wedge \cdots \wedge v_d$$

Proof. We write A as product of elementary $d \times d$ matrices:$A = E_m E_{m-1} \cdots E_1$. It is straightforward to check that $E_i u_1 \wedge \cdots \wedge E_i u_d = \det(E_i) \cdot u_1 \wedge \cdots \wedge u_d$. Therefore the statement follows. □

3 Varieties

Let us make our setting. Throughout the final project k will be an algebraically closed field. In first two sections we define affine and projective varieties. We often refer to affine or projective variety as just variety, usually it is clear from context what object should be taken into consideration.

3.1 Affine Space and Affine Varieties

Definition 3.1. An affine space of dimension n is the vector space k^n. We usually denote it by $\mathbb{A}^n$.

In affine space we will calculate values of polynomials on points. We calculate value of $f \in k[z_1, z_2, \dots, z_n]$ in point $\mathbf{x} = (x_1, x_2, \dots, x_n)$ by the evaluation homomorphism $e_{\mathbf{x}} : k[z_1, z_2, \dots, z_n] \to k$ (substitute each z_i with x_i and calculate inside the k). Note the main distinction between a vector space and an affine space, is that in affine setting we are allowed to translate the whole space for a fixed vector, in that way all points are "equally important". In the vector space point $\mathbf{0}$ plays a special role.

Definition 3.2. Let $S = \{f_i\}_{i\in I}$ be a collection of polynomials in $k[z_1, z_2, \dots, z_n]$. By $V(S)$ we denote the set of common zeros of polynomials in S: $V(S) = \{\mathbf{x} \in \mathbb{A}^n | f_i(\mathbf{x}) = 0, \forall i \in I\}$. It is the zero set of S, and if S is singleton then $V(S)$ will be called the zero set of the corresponding polynomial. A subset X of $\mathbb{A}^n$ is affine variety if there exists $S \subset k[z_1, z_2, \dots, z_n]$ such that $X = V(S)$.

Let us see some basic examples:

Example 3.3. Let $n = 1$. The affine space is just $k \cong \mathbb{A}^1$. Affine varieties are empty set, finite sets of points and the whole set $\mathbb{A}^1$. Empty set corresponds to $S = \{0\}$). A finite non-empty set $\{x_1, x_2, \dots, x_n\}$ is the zero set of polynomial $(x-x_1)(x-x_2)\cdots(x-x_n)$. $\mathbb{A}^1$ is also an affine variety, since $\mathbb{A}^1 = V(S)$ where $S = \{0\}$). Also no variety containing infinitely many points distinct from the $\mathbb{A}^1$ exists since any non-constant polynomial has just finitely many zeros.

Example 3.4. For $n = 2$, we have more affine varieties. Empty set is again an affine variety, since it is the zero set of polynomial 1. Singletons are affine varieties as well since $\{(a,b)\} = V(\{x-a, y-b\})$. In general, finite sets are affine varieties, but the checking we will leave for later (explained in 3.16). The zero sets of collections of form $\{y-ax-b\}$ or lines are affine varieties as well. Another (but very different) variety is given by $V(xy)$.

By 2.7 k is an infinite field, lines have infinitely many points. But in fact we don't get any new object this way (as our intuition says), but explanation what does it actually mean will wait till the next chapter.

Remark 3.5. Note that given any non-constant polynomial $p \in k[x,y]$ its zero set is non-empty, write it as $a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_1(x)y + a_0(x)$, and notice that if some $a_i(x) \neq 0$, for $i \geq 1$, then for some x_0 (actually, for an infinite number of them), $p(x_0, y)$ will be non-constant polynomial in y with zeros in the algebraically closed field. If, however, all $a_i(x)$ are zero, for $i \geq 1$ then $a_0(x)$ must be non-constant polynomial in x with zeros in the algebraically closed field.

3.2 Projective Space and Projective Varieties

In this chapter we introduce projective spaces and projective varieties.

Definition 3.6. By projective space over k we mean the set of one-dimensional subspaces of the vector space k^{n+1}. We denote it by $\mathbb{P}^n$.

If we don't specify an isomorphism of V with k^{n+1} for its projective space we write $\mathbb{P}V$. In other words we think of lines through the origin as of points. We introduce homogeneous coordinates: for line spanned by $z = (z_0, z_1, \ldots, z_n)$ we write $[z_0, z_1, \ldots, z_n]$, and for vector $v \in V$ the corresponding point in $\mathbb{P}V$ we denote by $[v]$.

Defining projective varieties is more trickier than in the affine case. The problem is that most of the polynomials do not define function on the projective space. However, we can talk about zero sets, without having well defined function. We should only have unambiguous notion of the zero set. The following theorem extracts such polynomials:

Theorem 3.7. *Let f be a polynomial in $k[z_0, z_1, \ldots, z_n]$. f induces well defined function $\psi : \mathbb{P}^n \to k$ given by $\psi([\boldsymbol{z}]) = 1 - \chi_0(e_{\boldsymbol{z}}(f))$ if and only if f is homogeneous, that is all of its monomials have the same degree, where χ_0 is defined by*

$$\chi_0(s) := \begin{cases} 1, & \textit{if } s = 0 \\ 0, & \textit{else} \end{cases}$$

Proof. Suppose f is homogeneous. Then obviously

$$f(\lambda z_0, \lambda z_1, \ldots, \lambda z_n) = \lambda^n \cdot f(z_0, z_1, \ldots, z_n),$$

where $\lambda \neq 0$, and LHS is zero iff RHS is zero. Therefore, a homogeneous polynomial vanishes either on the entire line which passes through the origin, or it doesn't vanish on any of its nonzero points. So $\psi([\mathbf{z}]) = 0$ if f vanishes on the line $\lambda\mathbf{z} \subset k^{n+1}$ or 0 if it doesn't vanish.

Assume f is non-homogeneous polynomial. We claim there exists a line passing through the origin containing two nonzero points $\mathbf{x}$ and $\mathbf{y}$ on this line such that f vanishes on $\mathbf{x}$, but does not vanish on $\mathbf{y}$.

Write f as the sum of its nonzero homogeneous pieces:

$$f = \sum_{i=1}^{n} g_i, \quad \deg g_i < \deg g_{i+1}$$

Since f is non-homogeneous, $n \geq 2$.

We prove there exists a point $\mathbf{z}$ where none of g_i vanishes. Suppose on the contrary that $k^{n+1} = \bigcup_{i=1}^{n} V(g_i)$. If g_1 is constant, remove $V(g_1)$ from the union, as it is the empty set the new union will still be the entire k^{n+1}. Remaining g_i cannot generate $k[x_1, x_2, \ldots, x_n]$ since 1 is not a linear combination of homogeneous pieces with deg ≥ 1. There exists a maximal ideal $\mathfrak{m}$ containing all of them by 2.13 and all of them vanish on the $V(\mathfrak{m})$ which is nonempty (by the Nullstellensatz, $I(V(\mathfrak{m})) = \mathfrak{m} \subsetneq k[x_1, \ldots, x_{n+1}] = I(\emptyset))$, a contradiction. Therefore there exists a point $\mathbf{z}$ we were looking for.

Consider function $p : k \to k$ defined by $p(\lambda) := f(\lambda\mathbf{z})$. It's not hard to see that $p \in k[\lambda]$ and that since $n \geq 2$, p is not monomial. Since k is algebraically closed, there exists $\lambda_0 \neq 0$ which is zero of p. For our $\mathbf{x}$ we choose $\lambda_0\mathbf{z}$. Since p is not a constant polynomial, there exists λ_1 such that $\lambda_1\mathbf{z}$ is not zero of p.

Since $[\mathbf{x}] = [\mathbf{y}]$, $\chi_0(e_{\mathbf{x}}(f) \neq \chi_0(e_{\mathbf{y}}(f))$, a contradiction and ψ cannot be well defined in this case. □

Remark 3.8. If k is not algebraically closed then we may have non-homogeneous polynomials for which ψ is well defined. Take $k = \mathbb{R}$ and $n = 1$, in k^2 polynomial $x^2 + y^2 + 1$ doesn't have zeros, so ψ is everywhere 1.

Definition 3.9. Let $S = \{f_i\}_{i \in I}$ be a collection of homogeneous polynomials in $k[z_0, z_1, \ldots, z_n]$. By $V(S)$ we denote the set of points in $\mathbb{P}^n$ where every polynomial in S vanishes. A subset $X \subset \mathbb{P}^n$ is called projective variety if there exists set of homogeneous polynomials S in $k[z_0, z_1, \ldots, z_n]$ such that $X = V(S)$.

Let us present some examples.

Example 3.10. Point $[a_0, a_1, \ldots, a_n]$ is a projective variety. At least one a_i is nonzero, and we can see that $[a_0, a_1, \ldots, a_n] = V(\{a_i z_0 - a_0 z_i, a_i z_1 - a_1 z_i, \ldots, a_i z_n - a_n z_i\})$.

The whole space is a projective variety, it is the zero set of homogeneous polynomial

0, and the empty set is variety, because it is the zero set of the collection of all homogeneous polynomials.

Example 3.11. We will show that finite subset Γ of $\mathbb{P}^n$ is a variety. Let $\Gamma = \{p_1, p_2, \ldots, p_m\}$. For $z \in \mathbb{P}^n$ and $q \in \mathbb{P}^n \setminus \Gamma$ we define $L_{i,q}(z) = \mathbf{t} \cdot \mathbf{z}$, where $\mathbf{t} \in k^{n+1}$ is such that $\mathbf{t} \perp \mathbf{p}_i$ and $\mathbf{t} \not\perp \mathbf{q}$, where $\mathbf{p}_i$, $\mathbf{z}$ and $\mathbf{q}$ are vectors in k^{n+1} corresponding to points z, p_i and q. $L_{i,q}$ is a linear homogeneous polynomial, vanishing on p_i, but not on the q. Define $L_q = L_{1,q} L_{2,q} \cdots L_{m,q}$. Γ is the zero set of the collection $\{L_q\}_{q \in \mathbb{P}^n \setminus \Gamma}$.

As we have said lines going through the zero in a vector space become points in the corresponding projective space. The lines in the projective space are obtained similarly: thus are planes going through the origin in the vector space, and we preserve incidence: a point is on a line iff the corresponding line was in the corresponding plane.

Example 3.12. Let v, w be vectors in V then $[v]$ and $[w]$ are corresponding points in the projective space. Lines going through origin containing v and w, respectively, are given by λv and μw and the plane determined by them is given by $\lambda v + \mu w$ for λ, μ in k. Therefore the corresponding line consists of all points $[\lambda v + \mu w]$. It is indeed an example of projective variety since it is the zero set of a homogeneous linear polynomial (we use those of the corresponding plane in the vector space).

Example 3.13. This generalizes previous example. Given vector space W over k of dimension $m+1$, embedded in V of dimension $n+1$. The inclusion gives rise to a map $\mathbb{P}W \hookrightarrow \mathbb{P}V$. The image of $\mathbb{P}W$ under such map is called a linear subspace of dimension m. If $m = 1$ then the image is line, and if $m = n-1$ then we call it hyperplane.
Linear space is a projective variety since we can find corresponding linear homogeneous forms (they have the form of the equations of linear spaces in the vector space).

Now we explain some relations between projective and affine space and projective and affine varieties. Let $U_i \subset \mathbb{P}^n$, for which $z_i \neq 0$. We can see that U_i is in fact an affine space because of the existence of natural bijection

$$[z_0, z_1, \ldots, z_n] \leftrightarrow (z_0/z_i, z_1/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_i).$$

Take U_0. Since $z_0 = 0$ determines a linear subspace, by the example 3.13, it also determines a projective subspace of $\mathbb{P}^n$. We decomposed $\mathbb{P}^n$ as $U_0 \sqcup \mathbb{P}^n \cong \mathbb{A}^n \sqcup \mathbb{P}^n$. Continuing in this manner, we obtain decomposition of $\mathbb{P}^n$:

$$\mathbb{P}^n = \coprod_{i=0}^{n} \mathbb{A}^i,$$

where $\mathbb{A}^0$ is just a point.
Another important observation is that projective space $\mathbb{P}^n$ is union of $n+1$ affine spaces $\mathbb{A}^n$. Moreover, any projective variety $X \subset \mathbb{P}^n$ is a union of $n+1$ affine varieties which can be embedded in $\mathbb{A}^n$. All we need to check is that given a projective variety X then $X \cap U_i$ is an affine variety. This is true since if X is determined by collection $\{F_\alpha\}_\alpha$ then $X \cap U_i$ is determined by collection

$$\{F_\alpha(z_1, z_2, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n)\}_\alpha.$$

Reversing process: given affine variety $X_0 \subset \mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$ we can find projective variety X such that intersection of U_0 with X is X_0: if X_0 is given by collection $\{f_\alpha\}_\alpha$ where

$$f_\alpha(z_1, z_2, \dots, z_n) = \sum a_{i_1,\dots,i_n} \cdot z_1^{i_1} \cdots z_n^{i_n}$$

of degree d_α, then X could be defined by $\{F_\alpha\}_\alpha$, where

$$F_\alpha(z_0, z_1, \dots, z_n) = z_0^{d_\alpha} f_\alpha(z_1/z_0, \dots, z_n/z_0) = \sum a_{i_1,\dots,i_n} \cdot z_0^{d_\alpha - \sum i_l} \cdot z_1^{i_1} \cdot \dots \cdot z_n^{i_n}.$$

Example 3.14. An example of decomposition of projective variety into disjoint affine varieties: given a projective line in projective plane, defined by $z_2 = z_0 + z_1$. Its homogeneous coordinates are $[x, y, x+y]$. We can see that, as a set, it consists of the line $[1, \frac{y}{x}, \frac{y}{x}+1]$ and "a point at infinity" $[0, 1, 1]$. Note that this is the decomposition (affine line+point) of any projective line, which is not surprisingly, since $\mathbb{P}^1 = \mathbb{A}^0 \coprod \mathbb{A}^1$

Example 3.15. Given affine line $\{(t, t+1) | t \in k\}$, described by $z_2 - z_1 = 1$, it can be embedded in U_0 as $\{[1, t, t+1] | t \in k\}$. By reconstruction we had, we see that corresponding projective variety can be given by $F_\alpha(z_0, z_1, z_2) = z_0(z_2 - z_1) - z_0^2$.

Example 3.16. Consider a finite set of points $\{\mathbf{x}_1, \dots \mathbf{x}_n\}$ and embed it in $U_0 \subset \mathbb{P}^n$ as $\mathbf{x}_i \to [1, x_{i1}, \dots, x_{in}]$. We have seen that finite set of points in $\mathbb{P}^n$ is a variety therefore its intersection with U_0 is also a variety, explaining claim given in 3.4.

A hypersurface is projective variety obtained as the zero set of single homogeneous polynomial.

Example 3.17. There are many examples. Take z_0 and we get projective line inside projective plane.
Another example is $V(z_0 z_3 - z_1 z_2)$. Its coordinates can be written as $[x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1]$ where $[x_0, x_1]$ and $[y_0, y_1]$ are projective lines (so not both x_0 and x_1 are zero, nor both y_0 and y_1 are zero). It is an example of so called Segre variety.

Remark 3.18. Note that if k is not algebraically closed, zero set of a homogeneous polynomial could consist of only one point, since e.g. $x_0^2 + y_0^2 = 0$ has only one solution in $\mathbb{R}$.

Example 3.19. Up to now we have seen only varieties that are hypersurfaces, linear spaces or finite sets of points. Here we present a new object, the twisted cubic. It is defined to be the image C of the map $v : \mathbb{P}^1 \to \mathbb{P}^3$ such that

$$v([x_0, x_1]) := [x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3].$$

We shall check that this is indeed a projective variety. Consider homogeneous polynomials in $k[z_0, z_1, z_2, z_3]$ given by the following equations:

$$f(z_0, z_1, z_2, z_3) = z_0 z_2 - z_1^2,$$

$$g(z_0, z_1, z_2, z_3) = z_0 z_3 - z_1 z_2,$$

$$h(z_0, z_1, z_2, z_3) = z_1 z_3 - z_2^2.$$

Let us prove $C = V((f, g, h))$ One side is obvious $C \subset V((f, g, h))$. Let $p = [x_0, x_1, x_2, x_3] \in V((f, g, h))$. One of the x_0, x_3 is nonzero since, otherwise, we must have also $x_1 = x_2 = 0$, which is impossible. If $x_0 \neq 0$ then $p = v([x_0, x_1])$, and if $x_3 \neq 0$ then $p = v([x_2, x_3])$. Another observation is that C is not the intersection of zero sets of any two polynomials of f, g, h. Intersections $V(f) \cap V(g)$, $V(g) \cap V(h)$ and $V(f) \cap V(h)$ contain lines $(0, 0, z_2, z_3)$, $(z_0, z_1, 0, 0)$ and $(z_0, 0, 0, z_3)$, respectively, which are not contained in C (in the next proposition, we see that no four points lie on the same plane in $\mathbb{P}^3$.

Proposition 3.20. *Any finite set of at least* 4 *points on a twisted cubic spans whole* $\mathbb{P}^3$.

Proof. We shall show that no four points from the twisted cubic lie on a projective plane. A point from the cubic has coordinates $[x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3]$, if it is in the plane $a_0 z_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 = 0$ then $a_0 (\frac{x_0}{x_1})^3 + a_1 (\frac{x_0}{x_1})^2 + a_2 (\frac{x_0}{x_1}) + a_3 = 0$ (we supposed $x_1 \neq 0$, analogously if $x_0 \neq 0$). But, there are at most three different solutions in k to this polynomial equation, therefore there are at most 3 points from cubic on a given plane. □

3.3 Subvarieties

Subvarieties of a variety X will just be the vanishing sets of polynomials in X:

Definition 3.21. Given affine variety X, a subset $Y \subset X$ is an affine subvariety of X, if there exists collection of polynomials such their common set of zeros is precisely Y.

Obviously, condition can be restated that $Y \subset X$ is its subvariety if there exists another variety Z such that $Y = X \cap Z$.

Definition 3.22. Given projective variety X, a subset $Y \subset X$ is called a projective subvariety of X, if there exists collection of homogeneous polynomials such their common set of zeros is precisely Y.

3.4 Zariski Topology

In this section we will define a particular topology on affine and projective spaces.

Definition 3.23. Let $X \subset \mathbb{A}^n$ be an affine variety. We endow it with the smallest topology such that each subvariety is closed. The resulting topology is called Zariski topology.

Zariski topology is a very natural topology. We just want to make polynomials continuous functions, where in the target set the $\{0\}$ is closed, which happens quite often (e.g., if the space is T_1). Since $\{0\}$ is closed, its preimage with respect to any polynomial, i.e. zero sets must be closed. If k is identified with $\mathbb{A}^1$, then the polynomials are indeed continuous.
Let us check that closed sets are exactly subvarieties. Firstly, the empty set and the whole space are closed, since they are the zero sets of polynomials 1 and 0, respectively. Therefore given variety X, the empty subset is again closed, but also the whole variety ($X = X \cap \mathbb{A}^n$). Given two affine varieties A_1 and A_2, generated by two collections of polynomials $\{p_i\}_i$ and $\{q_j\}_j$, their union is an affine variety again: $A_1 \cup A_2$ is the zero set of the collection $\{p_i q_j\}_{i,j}$: any element in the union is annihilated either by p_i or q_j, therefore surely by $p_i q_j$ and any element not annihilated by all p_i-s, nor all q_j-s, is not annihilated by some product $p_i q_j$. Now since $(A_1 \cup A_2) \cap X = (A_1 \cap X) \cup (A_2 \cap X)$ we have the same result for arbitrary variety. Given a collection of affine varieties $\mathcal{A}$ and their corresponding collections p_i, then $\bigcap A_{A \in \mathcal{A}}$ is the zero set of $\bigcup p_i$: any element of the intersection is annihilated by any polynomial in all collections, and if it is annihilated by all polynomials in all collections then obviously it is in the intersection. If $\{A_i \cap X\}_i$ are closed in X then $\bigcap_i (A_i \cap X) = (\bigcap_i A_i) \cap X$ is closed as well.

Remark 3.24. Given $S \subset k[z_1, z_2, \ldots, z_n]$ we observe that $V(S) = V(I(S)) = V(\mathrm{rad}(I(S))$, where $I(S)$ is the ideal generated by S, and $\mathrm{rad}(I(S))$ is a radical ideal containing $I(S)$).

To see this we notice that if $f(\mathbf{x}) = 0$ then $\forall g \in k[z_1, z_2, \ldots, z_n] : (g \cdot f)(\mathbf{x}) = 0$, and if $f(\mathbf{x}) = 0, g(\mathbf{x}) = 0$ then $(f+g)(\mathbf{x}) = 0$. The second equality holds, because if $f^n(\mathbf{x}) = f(\mathbf{x})^n = 0$ then $f(\mathbf{x}) = 0$.

Proposition 3.25. *A basis for the topology on an affine variety X is given by collection of sets $U_f = \{p \in X | f(p) \neq 0\}$ where f runs over all polynomials. Sets U_f are called distinguished open sets.*

Proof. We should prove that intersection of two distinguished open sets contains another distinguished open set, and that any open set is union of distinguished open sets. For the first condition we observe that intersection of two distinguished open sets U_f and U_g is again a distinguished open set: $U_f \cap U_g = U_{fg}$.
Now, given open subset U of X, then $U = X \cap V^c$ where $V = V(\mathfrak{a})$ is a variety. Note that $V^c \cap X = \bigcup_{f \in \mathfrak{a}} U_f$, since union is inside $V^c \cap X$ and for the other inclusion it is enough to see that if $\mathbf{x} \in V^c \cap X$ then for at least one $f \in \mathfrak{a}$, $f(\mathbf{x}) \neq 0$. □

Let us switch our attention now on projective varieties. The Zariski topology on a projective varieties is defined the same way.

Definition 3.26. Let X be a projective variety. Declare subvarieties of X to be closed. We obtain a topology on X called Zariski topology.

Again, we have to check that we indeed have topology and we do it the same way we did in the affine case. Empty set, and the whole variety are obviously closed: they are the zero sets of homogeneous polynomials 1, and 0.
Given two closed sets T_1 and T_2, which are the zero sets of homogeneous collections $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J_i}$, their union is closed set, it is the zero set of collection of homogeneous polynomials $\{p_i q_j\}_{i \in I, j \in J}$.
Given collection of closed sets T_i, which are the zero sets of homogeneous collections $\{p_{ij}\}_{j \in J_i}$, the intersection $\bigcap T_i$ is closed sets since it is the zero set of the homogeneous collection $\bigcup \{p_{ij}\}_{j \in J_i}$.

Definition 3.27. We call topological spaces which satisfy descending chain condition on closed sets, that is every sequence of its closed sets of form $Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset \cdots$ is stationary, Noetherian topological spaces.

Theorem 3.28. *Let X be a variety (affine or projective). Then, X is a Noetherian topological space.*

Proof. Consider an infinite sequence of closed subsets $Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset \cdots$. Obviously, $I(Z_0) \subset I(Z_1) \subset \cdots \subset I(Z_n) \subset \cdots$. Since every ideal is finitely generated (by Hilbert's Basis Theorem), the sequence must be stationary at some point. Therefore the sequence of closed subsets is also stationary. □

An important topological property of affine variety X is compactness, it easily follows from previous theorem:

Corollary 3.29. *Let X be a variety (affine or projective). Then X is Zariski-compact.*

Proof. Consider collection of closed subsets of X, $\{Z_i\}_{i\in I}$, such that $\bigcap_{i\in I} Z_i = \emptyset$. We should find finite subcollection, such that intersection remains empty set.
Take some set Z_1. If Z_1 is empty, we are done. If not, there exists Z_2, such that $Z_1 \supsetneq Z_1 \cap Z_2$, since the intersection of the whole collection is strictly contained in Z_1. Continue: if $Z_1 \cap Z_2 = \emptyset$, we are done, if not, there exists Z_3 such that $Z_1 \cap Z_2 \supsetneq Z_1 \cap Z_2 \cap Z_3$.
We inductively continue procedure, by the last statement it is not possible that inclusion is always strict, therefore for the some n we will get $Z_1 \cap \cdots \cap Z_n = \emptyset$, and this is finite subcollection of $\{Z_i\}_{i\in i}$. □

We prove following stronger statement, which we will need later:

Lemma 3.30. *Let $U_f \subset X$ be a distinguished open set, where $X \subset \mathbb{A}^n$ is an affine variety. Then U_f is compact in Zariski-topology on X.*

Proof. We will show that if $Z_i = V(\mathfrak{a}_i)$ are closed in X and $\bigcap(Z_i \cap U_f) = (\bigcap Z_i) \cap U_f = \emptyset$, then we can find finite refinement of Z_i such the same holds.
From the discussion after 3.23 and 3.24 we have $V(\text{rad}(\sum \mathfrak{a}_i)) = V(\sum \mathfrak{a}_i) = V((\bigcup \mathfrak{a}_i)) = \bigcap Z_i \subset V((f)) = V(\text{rad}((f))$ which is equivalent to $\text{rad}(\sum \mathfrak{a}_i) \supset \text{rad}((f))$, since by the Nullstellensatz $\text{rad}(\sum \mathfrak{a}_i) = I(V(\text{rad}(\sum \mathfrak{a}_i))) \supset I(V(\text{rad}((f)))) = \text{rad}((f))$. Equivalently, $f \in \text{rad}(\sum \mathfrak{a}_i)$. We can conclude there exists an integer m and elements $a_1, a_2, \ldots, a_s$ such that $f^m = a_1 r_1 + a_2 r_2 + \cdots + a_s r_s$ for some ring elements $r_1, r_2, \ldots, r_s$. If $a_i \in \mathfrak{a}_i$ then $f \in \text{rad}(\sum_{i=1}^{s} \mathfrak{a}_i)$ and therefore $Z_1, Z_2, \ldots, Z_s$ is our finite refinement. □

3.5 Quasi-projective variety

Let us recall the notion of locally closed subset: a subset of topological space is locally closed if it is the intersection of an open and closed set.
Now, let us introduce the notion which generalizes both affine and projective varieties:

Definition 3.31. A locally closed subset Y of a projective variety X is called a quasi-projective variety.

In other words, it is an open subset in the Zariski topology on our variety.

Example 3.32. Of course a projective variety $X \subset \mathbb{P}^n$ is a quasi-projective variety, since $X = \mathbb{P}^n \cap X$.

An affine variety Y can be placed in $U_0 \subset \mathbb{P}^n$, and since by observation given after example 3.14 there exists a projective variety $X \subset \mathbb{P}^n$ such that $X \cap U_0 = Y$. Since X is closed and U_0 open, Y is a quasi-projective variety.

A quasi-projective variety inherits a structure of topological space, by usual subspace topology.

4 Maps and Irreducibility

In this chapter we will study maps between varieties and explain what are irreducible and reducible varieties.

4.1 Regular Functions

We still haven't explained when two varieties are isomorphic. In this section we develop notions of regular functions, which are used in distinguishing between varieties.
Let us firstly deal the case of affine variety.

Definition 4.1. Coordinate ring of X is

$$A(X) := k[z_1, z_2, \ldots, z_n]/I(X).$$

Definition 4.2. Let U be a Zariski-open subset of X. Function f defined on U is regular at p if there exists neighbourhood $O \subset U$ of p such that $\forall q \in O$ we have $f(q) = \dfrac{g(q)}{h(q)}$, for two polynomials g, h, $h(p) \neq 0$.

The following theorem explains what are regular functions on varieties:

Theorem 4.3. *The ring of functions regular at every point of X is isomorphic to the coordinate ring $A(X)$. More generally, given $f \in k[z_1, z_2, \ldots, z_n]$ then ring of regular functions on U_f is isomorphic with $A(X)[1/f]$.*

Remark 4.4. By $A(X)[1/f]$ we mean ring $A(X)$ localized at multiplicative subset $\{1, f, f^2, \ldots\}$ i.e. we study the ring where the elements are fractions g/f^n, for some $g \in A(X)$ and some $n \geq 0$.

Proof. Let g be a regular function on U_f.
For each point $p \in X$ choose a neighbourhood O_p such that $g = h_p/t_p$ on O_p. Each O_p is a union of distinguished open sets by 3.25. We obtain a covering of U_f, and since it is compact by 3.30, we can find finite sub-covering. Let $U_{f_1}, U_{f_2}, \ldots, U_{f_n}$ be sets from this sub-covering. Since $U_{f_i} \subset O_p$ for some p, on U_{f_i} we have $g = h_p/t_p$. Let us write $g = h_i/t_i$ on U_{f_i}.
We have $f_i t_i g = f_i h_i$ on whole U_f, since f_i is zero outside U_{f_i}. Note that on U_f there

is no common zero of polynomials $f_i t_i$. This means $V((f_i t_i)_i) \subset V((f))$, and by the Nullstellensatz $f \in \mathrm{rad}((f_i t_i)_i)$, i.e. there exists m such that

$$f^m = \sum l_i t_i f_i$$

with l_i polynomials. But now

$$f^m g = \sum l_i t_i f_i \cdot (h_i/t_i)$$

that is,

$$g = \frac{\sum l_i f_i}{f^m} \in A(X)[1/f].$$

Taking $f = 1$, we see that the ring of regular functions on X is $A(X)$. □

Corollary 4.5. *The only regular functions on the affine space are polynomials.*

Proof. $\mathbb{A}^n$ is an affine variety and

$$A(\mathbb{A}^n) = k[x_1, x_2, \ldots, x_n]/I(\mathbb{A}^n) = k[x_1, x_2, \ldots, x_n]/(0) \cong k[x_1, x_2, \ldots, x_n].$$

□

Note: we will often use term polynomials for elements of the coordinate ring. In fact they are polynomials modulo the ideal of the variety. Since any element of the ideal vanishes on the entire variety, our terminology won't be ambiguous.

Example 4.6. Let us determine the ring of regular functions on the complement of origin $\{(0,0)\}$ in $\mathbb{A}^2$.
Let f be a regular function. Its restriction to $U_x = \mathbb{A}^2 \setminus V(x)$ takes the form $g(x,y)/x^n$, for adequate $g \in k[x,y]$ and some integer $n \geq 0$, and if g is divisible by x then $n = 0$, since the ring of regular functions on U_x is localization $k[x,y][1/x]$ by 4.3. Analogously, restriction on $U_y = \mathbb{A}^2 \setminus V(y)$ is $h(x,y)/y^m$, for adequate $h \in k[x,y]$ and some integer $m \geq 0$, and if h is divisible by y, then $m = 0$. Therefore on the intersection $U_x \cap U_y = \mathbb{A}^2 \setminus V(xy)$, we have $y^m g(x,y) = x^n h(x,y)$. This implies that $n = m = 0$, $h = g$, but also $f = g$ and since $\mathbb{A}^2 \setminus V(xy)$ is dense in $\mathbb{A}^2 \setminus \{(0,0)\}$ we have that it is polynomial on entire subset (and in fact on the entire plane). Therefore the coordinate ring is $k[x,y]$.

Let X be now a projective variety. We define $I(X)$ to be the ideal of the polynomials vanishing on X.

Definition 4.7. A regular function on a quasi-projective variety $X \subset \mathbb{P}^n$ or more generally on an open subset $U \subset X$ is a function such its restrictions on $U \cap U_i$ are all regular functions, where $\{U_i\}$ is cover of $\mathbb{P}^n$ by open sets $U_i \cong \mathbb{A}^n$.

So we check if the function is regular on a quasi-projective variety if it is regular locally, i.e. on each of the affine parts.

4.2 Regular Maps

Here we define maps between varieties:

Definition 4.8. Given two varieties $X \subset \mathbb{A}^m, Y \subset \mathbb{A}^n$, a regular map from X to Y is an n-tuple of regular functions on X, such that the image of the n-tuple is contained in Y.

By 4.3 all regular functions on an affine variety are polynomials (modulo the generating ideal). Therefore a regular map is given by n-tuple of polynomials.

Definition 4.9. Two affine varieties X and Y are isomorphic if there exists regular maps $\eta : X \to Y$ and $\phi : Y \to X$, such that $\phi \circ \eta = \mathrm{id}|_X$, $\eta \circ \phi = \mathrm{id}|_Y$.

A regular map is necessarily continuous:

Theorem 4.10. *Let X and Y be affine varieties and $\phi : X \to Y$ a regular map. Then ϕ is Zariski-continuous.*

Proof. Let $Z \subset Y$ be a closed set defined by the ideal $(g_1, \ldots, g_r)$ (by Hilbert's Basis Theorem, we can find finite set of generators). Then $f^{-1}(Z)$ is the zero set of $\{g_1(f_1, \ldots, f_n), \ldots, g_r(f_1, \ldots, f_n)\}$, where f_i are polynomials constituents to ϕ. □

This means that if affine varieties are isomorphic they are homeomorphic as well.

Example 4.11. Affine varieties $V(x^2 + x)$ and $V(xy)$ are not isomorphic.
$V(x^2+x)$ is obviously not connected since $V(x^2+x) = V(x+1) \cup V(x)$, $V(x+1) \cap V(x) = \emptyset$. On the other side $V(xy)$ is connected. Namely, if this wasn't the case we could represent $V(xy) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ as union of disjoint non-empty closed sets. This implies $(\mathfrak{a} + \mathfrak{b}) = (xy)$, because (xy) is radical. So there are $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a - b = xy$. At least one of a, b vanishes at $(0, 0)$ therefore other one must as well. But this means that our closed sets $V(\mathfrak{a})$ and $V(\mathfrak{b})$ have intersection, a contradiction. Now $V(x^2 + x)$ and $V(xy)$ are not homeomorphic, therefore they are not isomorphic by theorem 4.10.

Let us generalize definition we gave, so we include all quasi-projective varieties. Firstly we define regular map from an arbitrary quasi-projective variety to affine space.

Definition 4.12. Let X be a quasi-projective variety. A map $\phi : X \to \mathbb{A}^n$ is regular if it is given by n-tuple of regular functions. If the image is contained in an affine variety $Y \subset \mathbb{A}^n$, then we say that ϕ is regular mapping of X to Y and write $\phi : X \to Y$.

For the projective varieties the following definition applies:

Definition 4.13. Let X be a quasi-projective variety. A map $\phi : X \to \mathbb{P}^n$ is regular if for each $U_i \cong \mathbb{A}^n$, restriction $\phi|_{\phi^{-1}(U_i)} : X \to U_i$ is regular. If the image is contained in quasi-projective variety Y we also say that $\phi : X \to Y$ is a regular map from X to Y.

Example 4.14. In this example we will define projections, an important class of regular maps. Take a point $p \in \mathbb{P}^n$ and give it coordinates $[0, 0, \ldots, 1]$. Consider a copy of $\mathbb{P}^{n-1}$ described by coordinates $[z_0, z_1, \ldots, z_n, 0]$. Each point in q in $\mathbb{P}^n$ we send to the intersection of line $\overline{pq}$ with $\mathbb{P}^{n-1}$, coordinates description $\pi_p([z_0, z_1, \ldots, z_n]) \to [z_0, z_1, \ldots, z_{n-1}]$.
If $X \subset \mathbb{P}^n$ is a projective variety, then $\pi_p(X) \subset \mathbb{P}^{n-1}$. Consider affine subset $U_1 \subset \mathbb{P}^{n-1}$. We claim $\pi_p|_{\pi_p^{-1}(U_1)}$ is a regular map. We have $\pi_p^{-1}(U_1)$ is just $U_1' \subset \mathbb{P}^n$, the set of points with nonzero first coordinates. But on this affine set π_p is an n-tuple of polynomials $\pi_p = (1, z_1, \ldots, z_n)$ therefore it is a regular map. Analogously it is checked for other U_i-s and therefore projections are indeed regular maps.

Observe that every point of X and Y has affine neighbours. Regularity of ϕ implies regularity of the restriction $\phi|_{N_p} : N_p \to N_{\phi(p)}$ where N_x is an affine neighbourhood of point x, sufficiently small that its image under ϕ is contained in the affine neighbourhood $N_{\phi(p)}$.

Proposition 4.15. *A regular map $\phi : X \to Y$ between quasi-projective varieties is continuous.*

Proof. It is enough to show that it is continuous locally. For every point $p \in X$ choose affine neighbourhoods N_p and $N_{\phi(N)}$. Now $\phi|_{N_p} : N_p \to N_{\phi(p)}$ is a regular map of affine sets, therefore it is continuous by 4.10. □

4.3 Irreducible varieties

Some varieties act as building blocks of all varieties. The idea is to decompose a complicate object into atomic ones. Some authors even require irreducibility in the definition of variety. (See, e.g. [6].)
Let us firstly see, what is irreducibility in general context of topological spaces:

Definition 4.16. Topological space T is said to be irreducible if it cannot be expressed as union of two proper closed subsets. Empty set is by definition reducible.

Equivalently, T is irreducible if any two non-empty open subsets intersect.

Definition 4.17. Let X be a quasi-projective variety. We say that X is irreducible if it is irreducible as topological space in its Zariski-topology.

We have already seen in 3.28 that $\mathbb{A}^n$ and $\mathbb{P}^n$ are Noetherian topological spaces. Having this fact and the following theorem we understand why every variety is expressible as union of irreducible subvarieties:

Theorem 4.18. *In a Noetherian topological space T, every non-empty closed subset Y can be expressed as a finite union $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$, where Y_i are irreducible and closed in Y. If we require $Y_i \not\supseteq Y_j$ for $i \neq j$, then Y_i are uniquely determined.*

Proof. (taken from [6])
Firstly, let us find one representation. Let Σ be the set off all closed subsets which are not finite union of irreducible closed subsets. Suppose Σ is non-empty. Since T is Noetherian, Σ has a minimal element with respect to inclusion. Otherwise, we can form an infinite chain of decreasing closed subsets, where each one is strictly contained in the one before. Take a minimal element Y in Σ. Then Y is not irreducible, else in Σ are those which cannot be represented as finite union of irreducibles. Thus we can write $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets of Y. Since Y is minimal, Y' and Y'' must be representable as finite union of their proper closed subsets. But this implies Y is representable as finite union of proper closed subsets as well, which is a contradiction. Removing some of this closed subsets we get representation of $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$, where $Y_i \not\supseteq Y_j$, for $i \neq j$.
Let us prove the uniqueness of our representation. Let $Y = Y_1' \cup Y_2' \cdots \cup Y_m'$ be another representation such that $Y_i \not\supseteq Y_j$, for $i \neq j$. Then $Y_1' \subset \bigcup_{i=1}^r Y_i$, so $Y_1' = \bigcup_{i=1}^r (Y_1' \cap Y_i)$. But Y_1' is irreducible, therefore $Y_1' \subset Y_i$, for some i, say $i = 1$. Analogously, $Y_1 \subset Y_1'$, using the condition we have on different closed sets, we must have $Y_1 = Y_1'$. Let $Z = \overline{Y \setminus Y_1}$. Since $Z = Y_2 \cup \cdots \cup Y_r = Y_2' \cup \cdots \cup Y_m'$, proceeding by induction, we obtain the uniqueness of Y_i. □

Corollary 4.19. *Let X be a quasi-projective variety. Then X is representable as finite union of irreducible quasi-projective varieties.*

Proof. By 3.28, $\mathbb{A}^n$ and $\mathbb{P}^n$ are Noetherian topological spaces. □

We still didn't give any examples, before doing it let us prove an important property for irreducible varieties:

Proposition 4.20. *Let X be a variety (affine or projective). X is irreducible $\iff$ I(X) is prime ideal.*

Proof. We show this in affine case, the projective one is analogous.
Let us show that if X is irreducible then $I(X)$ is prime. Suppose on the contrary, that there exists polynomials p, q such that $pq \in I(X)$, $p, q \notin I(X)$. That means that on every point of X, pq vanishes, and there exist points $x, y \in X$ such that p vanishes on

x and q does not, and q vanishes on y, p does not. Consider subvarieties $P = V(p) \cap X$, $Q = V(q) \cap Y$. Obviously, $P \cup Q = X$ and $P \not\supseteq Q$, $Q \not\supseteq P$. Since Q, P are varieties on its own, we have contradiction on irreducibility of X. Conversely, let $I(X)$ be a prime. Suppose X is reducible, $X = V_1 \cup V_2$, where V_1 and V_2 are proper subvarieties of X. Then $I(V_1) \supsetneq I(V)$ and $I(V_2) \supsetneq I(V)$. Let $p \in I(V_1) \setminus I(V)$ and $q \in I(V_2) \setminus I(V_1)$. Now, we observe that pq vanishes on entire variety X, therefore it is in $I(X)$, but neither of p nor q is in $I(X)$, a contradiction. □

Example 4.21. Affine line $\mathbb{A}^1$ is irreducible. Closed sets in $\mathbb{A}^1$ are finite sets, and since k is an infinite field, it is not a union of two closed sets.
More generally, an affine space $\mathbb{A}^n$ is an irreducible variety. Suppose it can be represented as union of two proper closed sets, say T_1 and T_2, with corresponding collections $\{p_j\}_{i \in I}$ and $\{q_i\}_{j \in J}$. Obviously, $T_1 \subset V(p), T_2 \subset V(q)$, where p and q are arbitrary nonzero polynomials from corresponding collections (which exists since T_1, T_2 are proper subsets of $\mathbb{A}^n$). We know $\mathbb{A}^n = V(p) \cup V(q) = V(pq)$, but this implies $pq = 0$. But expanding gives us that either p or q is zero, which is a contradiction. Not-surprisingly, proof is actually just the proof that $k[x_1, x_2, \ldots, x_n]$ is integral domain. The similar proof would work for projective spaces.

Example 4.22. Another example is twisted cubic. In 3.19 we have seen that it is the zero set of $I = (xz - y^2, xt - yz, yt - z^2)$. We prove I is prime. Consider ring homomorphism

$$\alpha : k[x, y, z, t] \to k[s, p],$$

$$\alpha(x) \to s^3, \qquad \alpha(y) \to s^2 p, \qquad \alpha(z) \to sp^2, \qquad \alpha(t) \to p^3.$$

We prove the kernel of homomorphism is I and this will imply I is prime by 2.12. It is easy to see that $I \subset \ker \alpha$ and for the other side we pick $f \in k[x, y, z, t]$ and write it in form

$$f = a_0(x, t) + a_1(x, t)y + a_2(x, t)z \pmod{I}$$

with $a_i \in k[x, t]$ (we firstly "change" all yz with xt, and then all y^2 with xz and finally all z^2 by xy). If $f \in \ker \alpha$, then

$$0 = a_0(s^3, p^3) + a_1(s^3, p^3)s^2 p + a_2(s^3, p^3)sp^2$$

which is possible iff $a_i = 0$ for $i \in \{0, 1, 2\}$. Therefore $\ker \alpha = I$ and so I is prime.
By the Nullstellensatz we have $I(V((xz - y^2, xt - yz, yt - z^2)) = \mathrm{rad}((xz - y^2, xt - yz, yt - z^2)) = I$, and since I is prime irreducibility of the twisted cubic follows by 4.20.

Example 4.23. A hypersurface can be reducible and irreducible. If X is a hypersurface, $X = V(p)$ where p is a homogeneous irreducible polynomial in $k[x_1, x_2, \ldots, x_{n+1}]$,

then X is irreducible since $I(X) = I(V((p))) = (p)$ and (p) is prime ideal.
However, if p is reducible, one can find q, r such that $p|qr$, and $\deg q \leq \deg r < \deg p$, the principal ideal (p) is not prime, since $qsr \in (p)$, $q, r \notin (p)$, where s is an arbitrary monomial of degree $\deg r - \deg q$. By 4.20 X is not irreducible variety.

Example 4.24. Another non-example is $V(xy) \subset \mathbb{A}^2$. It is reducible since $V(xy) = V(x) \cup V(y)$ (it is the union of x and y axis). Equivalently, ideal (xy) is not prime, $xy \in (xy)$, $x, y \notin (xy)$. As $V(x)$ and $V(y)$ are both irreducible varieties, and $V(x) \not\supseteq V(y), V(y) \not\supseteq V(x)$ we found the unique representation of $V(xy)$ from 4.19.

Proposition 4.25. *Let X and Y be a topological spaces and $f : X \to Y$ a continuous map. If X is irreducible, then so is $f(X)$ in the subspace topology.*

Proof. Suppose $f(X) = V_1 \cup V_2$, where V_1, V_2 are nonempty closed in Y, then $X = f^{-1}(V_1) \cup f^{-1}(V_2)$ with $f^{-1}(V_1)$ and $f^{-1}(V_2)$ closed in X. Since X is irreducible one of them, say $f^{-1}(V_1)$ is the empty set, the other is the whole X. But this implies $V_1 = \emptyset$, $V_2 = Y$ a contradiction. □

Corollary 4.26. *Let X and Y be quasi-projective varieties and $\phi : X \to Y$ a regular map. Then $\phi(X)$ is irreducible in the subspace topology.*

Proof. By 4.15 ϕ is continuous and the statement follows by the previous proposition. □

Example 4.27. Now we can see that some varieties are not isomorphic. For example $V(xy)$ is not isomorphic to the affine line as asserted in the 3.4.

We will need later the following lemma:

Lemma 4.28. *Let $X \subset \mathbb{P}^n$ be a projective variety and let $D_0, D_1, \ldots, D_s$ be open irreducible subsets of X such that $D_i \cap D_j$ is nonempty for all i, j. Then X is irreducible.*

Proof. Let $X = X_1 \cup X_2$ where X_1 and X_2 are closed. For each i we have $D_i = (D_i \cap X_1) \cup (D_i \cap X_2)$, so either $D_i = (D_i \cap X_1)$ or $D_i = (D_i \cap X_2)$. If $D_i \neq D_i \cap X_2$ then $D_i = D_i \subset X_1$. Therefore, X_1 contains $D_i \cap D_j$ for all j, which are open sets. □

5 Grassmannian Varieties

5.1 Grassmannian Varieties

We construct a projective space by taking the lines trough the origin in $V = k^n$ and identify them as points. A natural generalization is to take d-dimensional subspaces and consider them as points. We will denote by $G(d, n)$ the set of d-dimensional linear subspaces of k^n. If we don't specify basis we write $G(d, V)$, and since a d-dimensional subspaces is the same thing as a $d-1$-plane in the corresponding projective space $\mathbb{P}^{n-1}$, we can think of $G(d, n)$ as the set of such $d-1$ planes; when we think of Grassmannian in this way we write it as $G(d-1, n-1)$.
Let us see Grassmannians as a subsets of a projective space. Consider a Grassmannian $G(d, V)$. Associate to each d-dimensional subspace $W \subset V$ spanned by vectors $v_1, v_2, \ldots, v_d$ the vector

$$\lambda = v_1 \wedge v_2 \wedge \cdots \wedge v_d \in \wedge^d V.$$

By 2.27, choosing different basis for W would yield in multiplying λ by a determinant of the change basis matrix. Thus we have a well defined map

$$\psi : G(d, V) \to \mathbb{P}(\wedge^d V).$$

Proposition 5.1. $\psi : G(d, V) \to \mathbb{P}(\wedge^d(V))$ *is an injective function.*

Proof. Define $\phi : \psi(G(d, V)) \to G(d, V)$ as follows $\phi([w]) = \{v \in V | v \wedge w = 0 \in \wedge^{d+1} V\}$. Let us see that ϕ is well defined. Each $\phi([w])$ is a subspace of V and it doesn't depend on a particular choice of w. Let $[w] = [w_1 \wedge \cdots \wedge w_d] = \psi(W)$ where $w_1, w_2, \ldots, w_d$ is a basis for W. Since for any $v \in W$ we have $w \wedge v = 0$, $W \subset \phi(\psi(W))$. Now take any $v \in \phi(\psi(W))$, such that $v \wedge w_1 \wedge \cdots \wedge w_d = 0$. Extend w_i to basis of V with vectors $w_{d+1}, \ldots, w_n$. We write $v = \sum a_i w_i$ and we see that $(\sum a_i w_i) \wedge w_1 \wedge \cdots \wedge w_d = 0$ iff $a_i = 0$ for $i > d$ (since elements in form $w_1 \wedge \cdots \wedge w_d \wedge w_i$ are linearly independent by 2.24.) Therefore the image is always of dimension d and from proof it is evident that $\phi \circ \psi = id$ which means that ψ is injective. $\square$

From now on we call ϕ the Plücker embedding of $G(d, V)$.
The homogeneous coordinates on $\mathbb{P}^N = \mathbb{P}(\wedge^d V)$ (where $N = \binom{n}{d} - 1$) are called Plücker coordinates on $G(d, V)$. If $V = k^n$ we represent W by the $d \times n$ matrix M_W whose rows

are vectors w_i which span W; such matrix is determined up to multiplication on the left by an invertible $d \times d$ matrix. We can see Plücker coordinates as the determinants of maximal minors of M_W, since multiplying M_W by an invertible $d \times d$ matrix yields in multiplying each of determinants by the same constant.

Example 5.2. We will associate 6 coordinates to each line in $\mathbb{P}^3 = \mathbb{P}(k^4)$. Every line l in $\mathbb{P}^3$ corresponds to a plane W in k^4 containing the origin. Take two vectors v_1 and v_2, which span W. First row of M_W is v_1 and second row is v_2. Plücker coordinates are 6 determinants of 2×2 submatrices. If we multiply M_W by an invertible 2×2 matrix from the left, we see that every determinant is multiplied by the same constant, which actually yields same coordinates.

Lemma 5.3. *Consider the following matrix :*

$$T([\boldsymbol{z}]) := \begin{pmatrix} q_{1,1}(z_0,\dots,z_n) & q_{1,2}(z_0,\dots,z_n) & \cdots & q_{1,p}(z_0,\dots,z_n) \\ q_{2,1}(z_0,\dots,z_n) & q_{2,2}(z_0,\dots,z_n) & \cdots & q_{2,p}(z_0,\dots,z_n) \\ \vdots & \vdots & \cdots & \vdots \\ q_{s,1}(z_0,\dots,z_n) & q_{s,2}(z_0,\dots,z_n) & \cdots & q_{s,p}(z_0,\dots,z_n) \end{pmatrix},$$

where z_i are indeterminates and $q_{i,j}$ are homogeneous linear polynomials in z_i. Then the rank of homogenized matrix $T(p)$, (T calculated at point $p = [p_0,\dots,p_n]$) is less or equal than m if and only if all minors $(m+1) \times (m+1)$ vanish.

Remark 5.4. Let us explain our terminology. Homogenized matrix is just a representative of the equivalence class $[M]$, where the relation is given $M \sim N$ iff there exists nonzero scalar λ such that $\lambda M = N$. We calculate value of our matrix at point p, by plugging p_i in the place of z_i. An $(m+1) \times (m+1)$ minor is the determinant of $(m+1) \times (m+1)$ submatrix. It is a homogeneous polynomial of degree $m+1$.

Proof. Let the rank of $T(p)$ be r. Then every $(r+1) \times (r+1)$ minor must vanish at p, because determinant of every $(r+1) \times (r+1)$ submatrix is zero.
Let $t < r$. Then there exists $(t+1) \times (t+1)$ submatrix with full rank. So corresponding minor doesn't vanish at p. □

We are ready to prove that indeed Grassmannians are projective varieties:

Theorem 5.5. *$G(d, V)$ is a projective variety.*

Proof. We will describe totally decomposable vectors i.e. vectors $w \in \wedge^d V$, that are products $w = v_1 \wedge \cdots \wedge v_d$, where each $v_i \in V$ (this is enough since in Plücker embedding totally decomposable vectors are exactly those which are in the Grassmannian). We have seen in 2.26 that a vector $w \in \wedge^d V$ is divisible by $v \in V$, i.e. $w = v \wedge \phi$ for some

$\phi \in \wedge^{d-1}V$, iff $w \wedge v = 0 \in \wedge^{d+1}V$. Now it follows that nonzero vector w is totally decomposable iff the space, spanned by the vectors dividing it, is d-dimensional. Thus, $[w]$ will lie in the Grassmannian if and only if the rank of the map

$$\phi(w) : V \to \wedge^{d+1}V$$

$$v \to w \wedge v$$

is $n-d$. Rank of $\phi(w)$ is never less than $n-d$, since the space of vectors dividing w is not of dimension more than d and therefore

$$[w] \in G(d,V) \iff \operatorname{rank}(\phi(w)) \leq n-d.$$

Map $\phi : \wedge^d V \to \operatorname{Hom}(V, \wedge^{d+1}V)$ sending w to $\phi(w)$ is linear, because

$$\begin{aligned}\phi(\lambda w' + \mu w'') &= v \longmapsto (\lambda w' + \mu w'') \wedge v \\ &= \lambda\phi(w') + \mu\phi(w'').\end{aligned}$$

Let $w = [w_0, \ldots, w_N]$ be coordinates of w. Because ϕ is linear, every entry of $\binom{n}{d+1} \times n$ matrix $\phi(w)$ must be a linear combination of $w_0, \ldots, w_N$. Therefore ϕ can be written as matrix Φ whose (i,j)-th entry is given by $\sum_{l=0}^{N} c_{ijl}z_l$, where z_l are indeterminates so that $\phi(w)$ substitutes w_l instead of z_l. Matrix $\phi(\lambda w)$ is just matrix $\phi(w)$ multiplied by λ. Therefore the "homogenized" matrices will be the same.
Now we note that $[w] \in G(d,V)$ iff $\operatorname{rank}(\phi(w)) \leq n-d$ which is equivalent to that that all $(n-d+1) \times (n-d+1)$ minors vanish, by 5.3. So a collection of polynomials that "cut off" $G(d,V)$ is given by $(n-d+1) \times (n-d+1)$ minors of matrix Φ. □

From now on we implicitly assume that Grassmannians are embedded by the Plücker embedding. The following observation about Grassmannians is crucial:

Lemma 5.6. *Let $G(d,n)$ be the Grassmannian embedded by the Plücker embedding in $\mathbb{P}(k^N)$ (where $N = \binom{n}{d}$). Let U_0 be the set of the elements of the projective space $\mathbb{P}(k^N)$ with nonzero first coordinate. Then $U_0' = U_0 \cap G(d,n)$ is naturally an affine variety and $U_0' \cong A^{d(n-d)}$ as affine varieties.*

Proof. Since U_0' is embedded into the affine subset U_0 it is naturally an affine variety. Let us prove the isomorphism.
To every point W of a $G(d,n)$ we associate corresponding matrix M_W, determined up to multiplication to the left by an invertible $d \times d$ matrix, as explained before. The matrix determines the Plücker coordinates. Since the first homogeneous coordinate of any point in U_0' is nonzero, we can choose it to be 1. This means that we can choose

our M_W to be of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & z_{1,d+1} & z_{1,d+2} & \cdots & z_{1,N} \\ 0 & 1 & \cdots & 0 & z_{2,d+1} & z_{2,d+2} & \cdots & z_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & z_{d,d+1} & z_{d,d+2} & \cdots & z_{d,N} \end{pmatrix}$$

(note that here by first coordinate we mean the coordinate associated with the matrix determined by first d columns).
Consider map $\phi : \mathbb{A}^{d(n-d)} \to \mathbb{P}^N$, such that

$$\phi((a_{i,j})_{i,j}) = [p_{i_1,\ldots,i_d}(a_{i,j})]_{1\leq i_1<\cdots<i_d\leq n}, \quad i = 1, 2, \ldots, d; j = 1, 2, \ldots, n-d,$$

where $p_{i_1,\cdots,i_d}$ is the determinant of $d \times d$ submatrix of M_W determined by the columns $i_1, \ldots, i_d$. The map ϕ is regular: it is given by the $\binom{n}{d}$ polynomials, and the first of which is 1, so the image is completely contained in U_0, which is naturally an affine space. Even more, the image of the map is exactly U_0', since for a point $S \in U_0'$ we choose M_S having the first $d \times d$ submatrix the identity matrix. Then M_S is the image under ϕ of the point corresponding to the submatrix obtained after removing the first $d \times d$ identity-submatrix. Map ψ which associates to S corresponding preimage under ϕ is obviously regular (it is a map that "just reads" some coordinates and "forgets" others). Thus $U_0' \cong \mathbb{A}^{d(n-d)}$ as affine varieties. □

5.1.1 Irreducibility of Grassmannian varieties

In this subsection we will prove an important property of Grassmannian varieties, their irreducibility.

Proposition 5.7. *Grassmannian variety $G(d, n)$ is irreducible.*

Proof. Embed the Grassmannian by the Plücker embedding into $\mathbb{P}(k^N)$, $N = \binom{n}{d}$. For $0 \leq i \leq N-1$ by U_i we denote the affine subset with i-th nonzero coordinate. Every i corresponds to a d-element subset of $\{1, \ldots, n\}$, we let $i = 0$ corresponds to $\{1, \ldots, d\}$. Define $U_i' = G(d, n) \cap U_i$.
We prove the intersection of any two U_i'-s is nonempty. Let us check it for U_0' and U_j', $j \geq 1$. Take $\{i_1, \ldots, i_d\}$ corresponding to j (where $1 \leq i_1 < \cdots < i_d \leq n$ and $\{i_1, \ldots, i_d\} \neq \{1, \ldots, d\}$). Consider a matrix M_W for which the first $d \times d$ submatrix is the identity matrix, and i_1-th, $\ldots, i_d$-th columns are such that determinant of the corresponding submatrix is 1, without touching first d columns. Now at least two $d \times d$ submatrices will have determinant 1, the one determined by columns $\{1, \ldots, d\}$, and the other determined by columns $\{i_1, \ldots, i_d\}$. This makes 0-th and j-th coordinate of

corresponding point equal to 1, so it is contained in both U'_0 and U'_j. By 4.28 we have that the $G(d, n)$ are irreducible. $\square$

6 Conclusion

In the final project paper we presented some notions and results from classical algebraic geometry.
Varieties are vanishing sets of polynomials. We studied them, taking ground field k to be algebraically closed. The main distinction is on affine and projective varieties. We have seen some relations between them. On affine and projective space we have defined useful topology: Zariski topology. It is obtained by declaring varieties and subvarieties as closed sets.
To distinguish varieties one needs notion of regular function and regular map. Regular function is a function that is locally quotient of two polynomials. On affine varieties, it turns out that regular functions are exactly polynomials. Regular map is given by an n-tuple of regular functions. If it has inverse map, that is also regular, then corresponding varieties are isomorphic.
Atomic varieties are called irreducible varieties. They cannot be represented as union of two proper subvarieties. Every variety can be represented as finite union of irreducible varieties.
An important class of varieties are Grassmannian varieties. Grassmannian variety $G(d, n)$ is the set of d-dimensional subspaces of $V = k^n$. They get natural structure of projective variety after embedding in $\mathbb{P}(\wedge^d V)$. As projective variety, $G(d, n)$ is irreducible. For further reading see: [1], [6]. Some of the proofs were taken from: [1], [5], [2], [6].

7 Povzetek naloge v slovenskem jeziku

Algebraična geometrija je ena od aktivnih raziskovalnih področij v matematiki. Cilj našega zaklučnega dela, je da predstavimo osnovne pojme in rezultate v klasični algebraični geometriji. Pri tem predpostavljamo, da bralec pozna osnovne pojme in rezultate iz linearine algebre, splošne topologije ter komutativne algebre.

Bodi k algebraično zaprt komutativen obseg. Z $\mathbb{A}^n = k^n$ označimo vektorski prostor dimenzije n nad obsegom k. Rečemo mu afini prostor dimenzije n. Bodi še S podmnožica množice $k[x_1, x_2, \ldots, x_n]$ vseh polinomov v n nedoločenkah $x_1, \ldots, x_n$. Vrednost polinoma $p \in k[x_1, x_2, \ldots, x_n]$ v točki $(a_1, a_2, \ldots, a_n) \in \mathbb{A}^n$ dobimo tako da zamenjamo nedoločenke x_i za a_i, ter dobljeni izraz izračunamo znotraj polja k. Z $Z(S)$ označimo množico vseh točk v k^n, kjer so vsi $p \in S$ zavzamejo ničelno vrednost. Podmnožici X množice $\mathbb{A}^n$, za katero obstaja taka kolekcija polinomov $S \subset k[x_1, x_2, \ldots, x_n]$ tako da $X = Z(S)$ rečemo algebraična raznoterost.
Projektivni prostor $\mathbb{P}^n$ dobimo iz vektorskega prostora k^{n+1} tako da za njegove točke proglasimo množico vseh enorazsežnih linearnih podprostorov v k^{n+1} (tj. vse premice, ki grejo skozi točko $(0, 0, \ldots, 0)$. V projektivnem prostoru je redkokdaj mogoče enolično ovrednotiti polinom v dani točki. V primeru homogenih polinomov lahko preverimo vsaj to, ali je v dani projektivni točki ničeln ali ne. Projektivna raznoterost je definirana kot množica projektivnih točk, kjer zavzamejo vsi homogeni polinomi iz neke družine ničelno vrednost .
Vsaki raznoterosti X (afini ali projektivni) lahko pridružimo ideal $I(X)$ vseh polinomov v $k[x_1, x_2, \ldots, x_n]$, ki na raznoterosti zavzamejo konstantno ničelno vrednost. V zaključni nalogi je pokazano da se vsak projektivni prostor lahko zapiše kot disjunktna unija afinih prostorov. Poleg tega lahko projektivni prostor $\mathbb{P}^n$ zapišemo tudi kot končno unijo afinih prostorov, ki so na naraven način kopije prostora $\mathbb{A}^n$.
Na afinih in projektivnih prostorih (še bolj splošno na raznoterosth) lahko vpeljemo zelo pomembno topologijo, imenovano topologija Zariskega. Po definiciji je to najmanjša topologija, kjer so vse raznoterosti na njej zaprte množice. Topologija Zariskega nam omogoča, da lahko preučujemo topološke lastnosti na raznoterosth, od katerih je še zlasti pomembna irecudibilnost.

Spomnimo, da je X ireducibilna raznoterost če je X ireducibilen topološki prostor (torej ni unija pravih nepraznih zaprtih podmnožic). Pomembna lastnost ireducibilnih raznoterost je da je v tem primeru njen ideal polinomov praideal.
Da bi lahko ločili raznoterosti, rabimo pojem regularne funkcije ter regularne preslikave. Regularna funkcija na raznoterosti X je funkcija ki je lokalno kvocient dveh polinomov (v primeru projektivne raznoterosti morata biti to homogena polinoma iste stopnje). Izkaže se, da so v primeru afine raznoterosti X regularne funkcije natanko vsi polinomi modulo ideal $I(X)$. Regularna preslikava raznoterosti X na afini prostor $\mathbb{A}^m$ je dana z m-terico polinomov. Regularna preslikava $\phi : X \to Y$ za afino raznoterost $Y \in \mathbb{A}^m$ je regularna preslikava na $\mathbb{A}^m$ katere slika leži v Y. Regularna preslikava $\phi : X \to Y$, kjer Y je projektivna raznoterost, je preslikava ki je lokalno regularna preslikava med afinimi raznoterostmi. Regularni bijektivni preslikavi, katere inverz je tudi regularen rečemo izomorfizem med raznoterostmi. Posebej so zanimive Grassmannian raznoterosti, $G(d, n)$, ki so definirane kot množica vseh d-razsežnih prostorov v k^n. A priori $G(d, n)$ nima strukture raznoterosti. Če pa uporabimo Pĺ uckerjevo vložitev, ki pošlje d-razsežni prostor W, z bazo $\{w_1, w_2, \dots, w_d\}$ v ekvivalenčni razred $[w_1 \wedge w_2 \wedge \cdots \wedge w_d] \in \mathbb{P}^{\binom{n}{d}-1} = \mathbb{P}(k^{\binom{n}{d}})$ dobimo množico za katero se izkaže da ima naravno strukturo projektivne raznoterosti. V zaključni nalogi je tudi pokazano, da je raznoterost $G(n, d)$ ireducibilna.

8 Bibliography

[1] J. Harris, *Algebraic Geometry - A First Course*, Springer Verlag, First Edition, 1992. *(Cited on pages 1, 4, and 31.)*

[2] S. Richelson, *Classifying Varieties with Many Lines*, Senior Thesis, Harvard University, 2008. *(Cited on page 31.)*

[3] M.F. Attiyah and I.G. Macdonald, *Introduction to Commutative Algebra.* Addison-Wesley Publishing Co., 1969. *(Cited on pages 4 and 5.)*

[4] S. Lang, *Algebra (Graduate Texts in Mathematics)*, Springer Verlag, Revised 3rd Edition, 2002. *(Cited on page 2.)*

[5] J. Tong, *Introduction to Algebraic Geometry*, 2012. *(Cited on page 31.)*

[6] R. Hartshorne, *Algebraic Geometry (Graduate Texts in Mathematics*, Springer Verlag, Corr. 8th Edition, 1997. *(Cited on pages 22, 23, and 31.)*

[7] P. Halmos, *Naive Set Theory*, Springer Science and Business Media, 1960. *(Cited on page 2.)*