Order Statistic Problems on Suffixes

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- *Generic multi-selection*: $\mathcal{R} \subset \{1, 2, \ldots, n\}$.

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- ... but results can be *easily extended* to *multidimensional objects*, like *strings* and *vectors*, and the *lexicographical order*.
- *Not so easy* when the input objects are the *suffixes of a sequence* T.
- In all cases, the *comparison model* is considered: the input objects (or for the cases of strings, vectors and suffixes, the elements they are made of) *can only be compared*.

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  - classic example of *divide et impera* approach.

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Natural question:

Are the complexities of Suffix Sorting and Suffix Selection the same?

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  - ...same sequence T but only *a fraction of the n suffixes* would be considered in the sub-problems.

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  - If the approach was applied to suffixes, the two *recursive subproblems* (the finding of the median of medians and the recursive application on $\mathcal{L}$ or $\mathcal{R}$) *would not be instances of the Suffix Selection problem* anymore...
  - ...same sequence T but only *a fraction of the n suffixes* would be considered in the sub-problems.
- However, we will use the selection algorithm in [Blum et al, JCSS 7, 1973] as a *basic tool for suffix selection*.

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Our knowledge about the k-th smallest suffix is increased during *Phase Transitions*.

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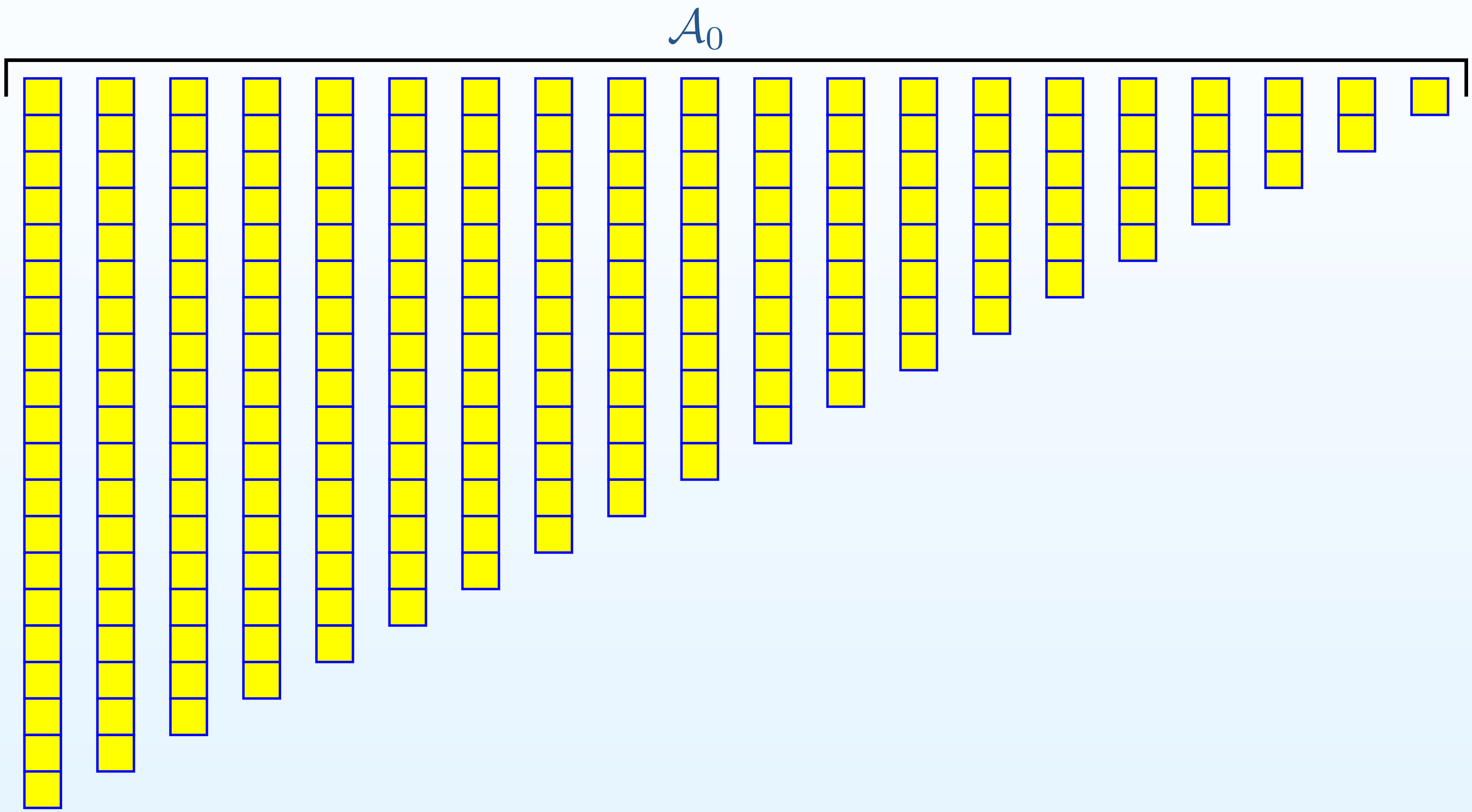
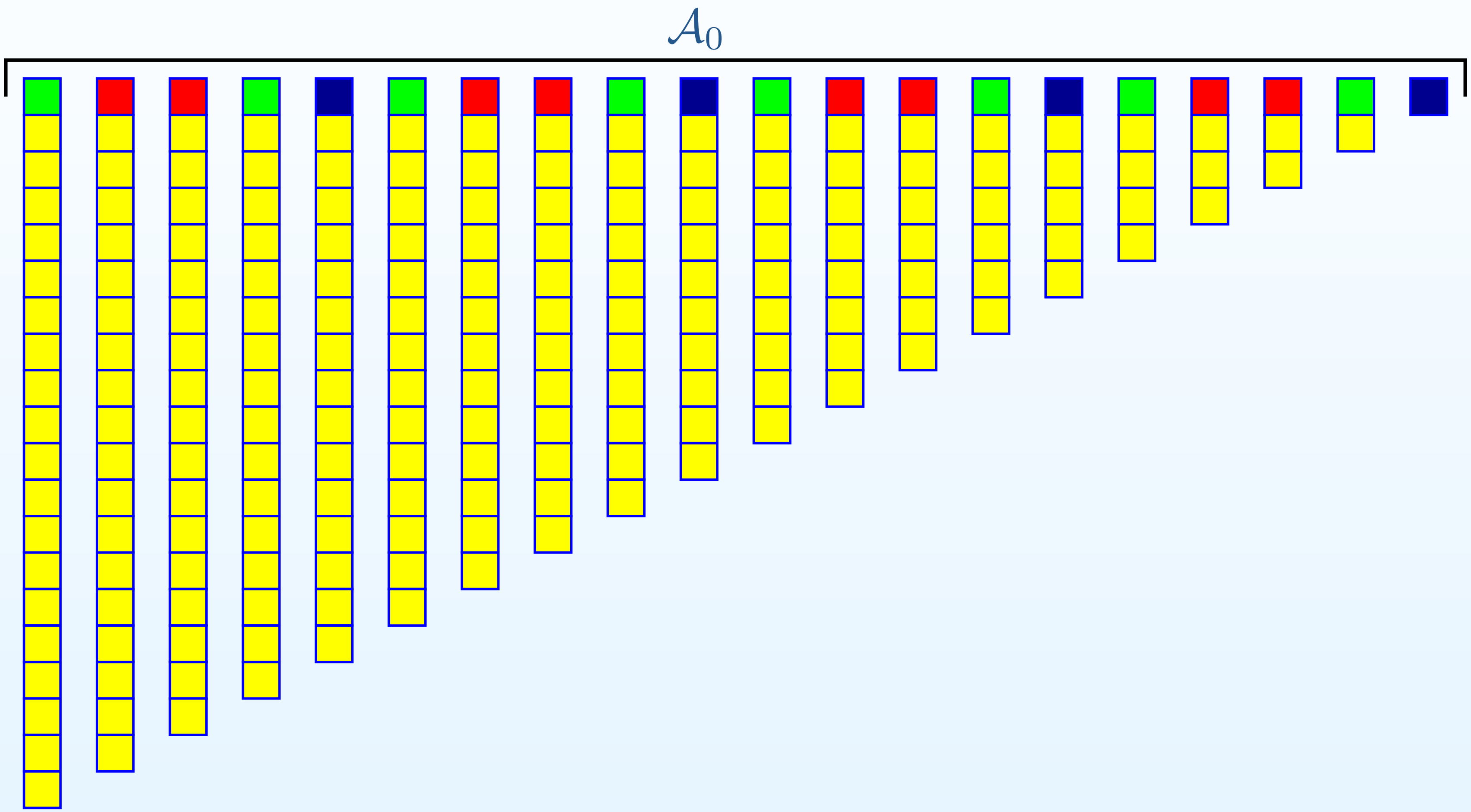
The computation ends when a phase transition leaves us *with only one active suffix*.

Suffix selection, first attempt



$k = 10$, $l_0 = 0$, Phase 0

Suffix selection, first attempt



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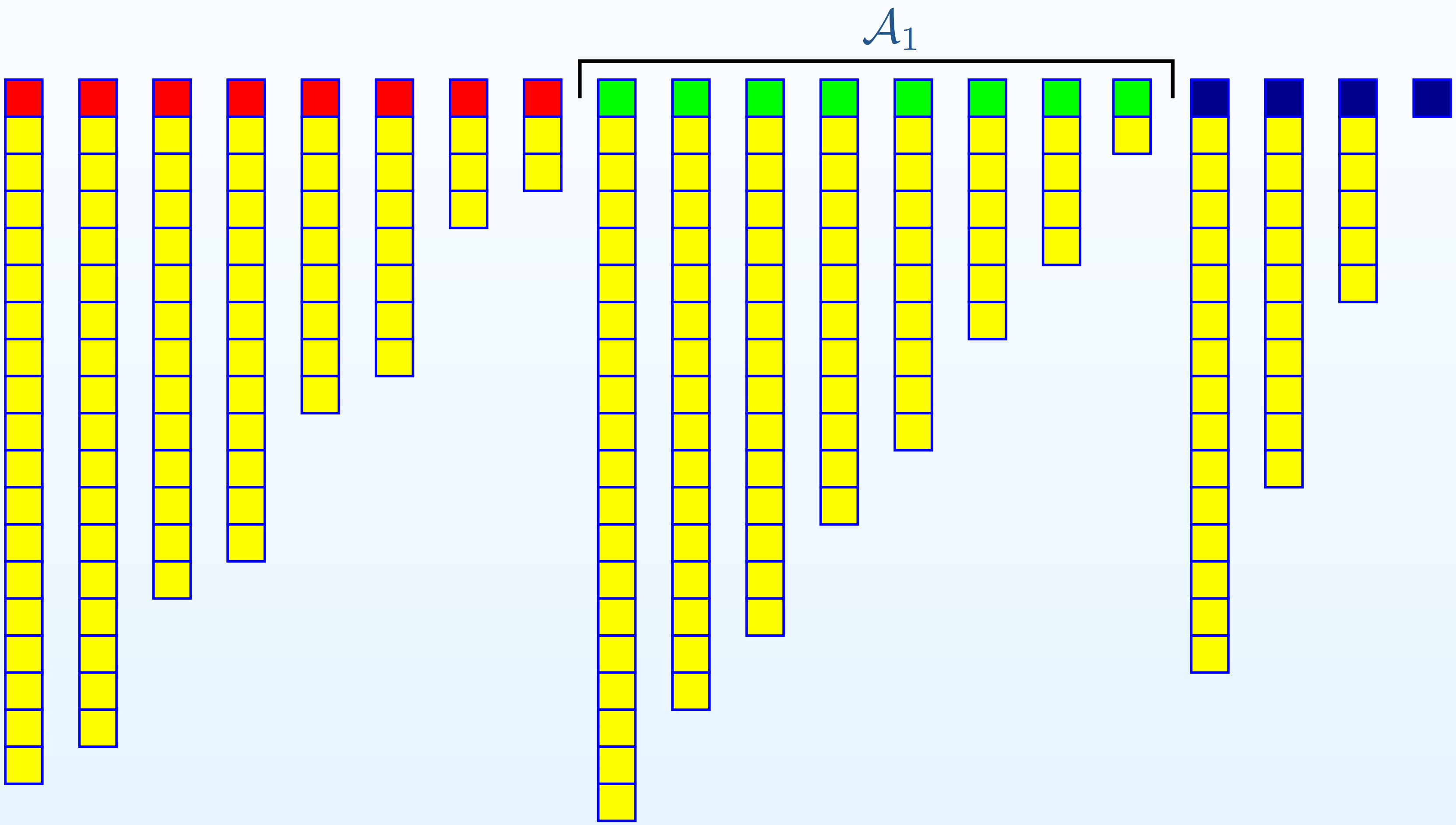
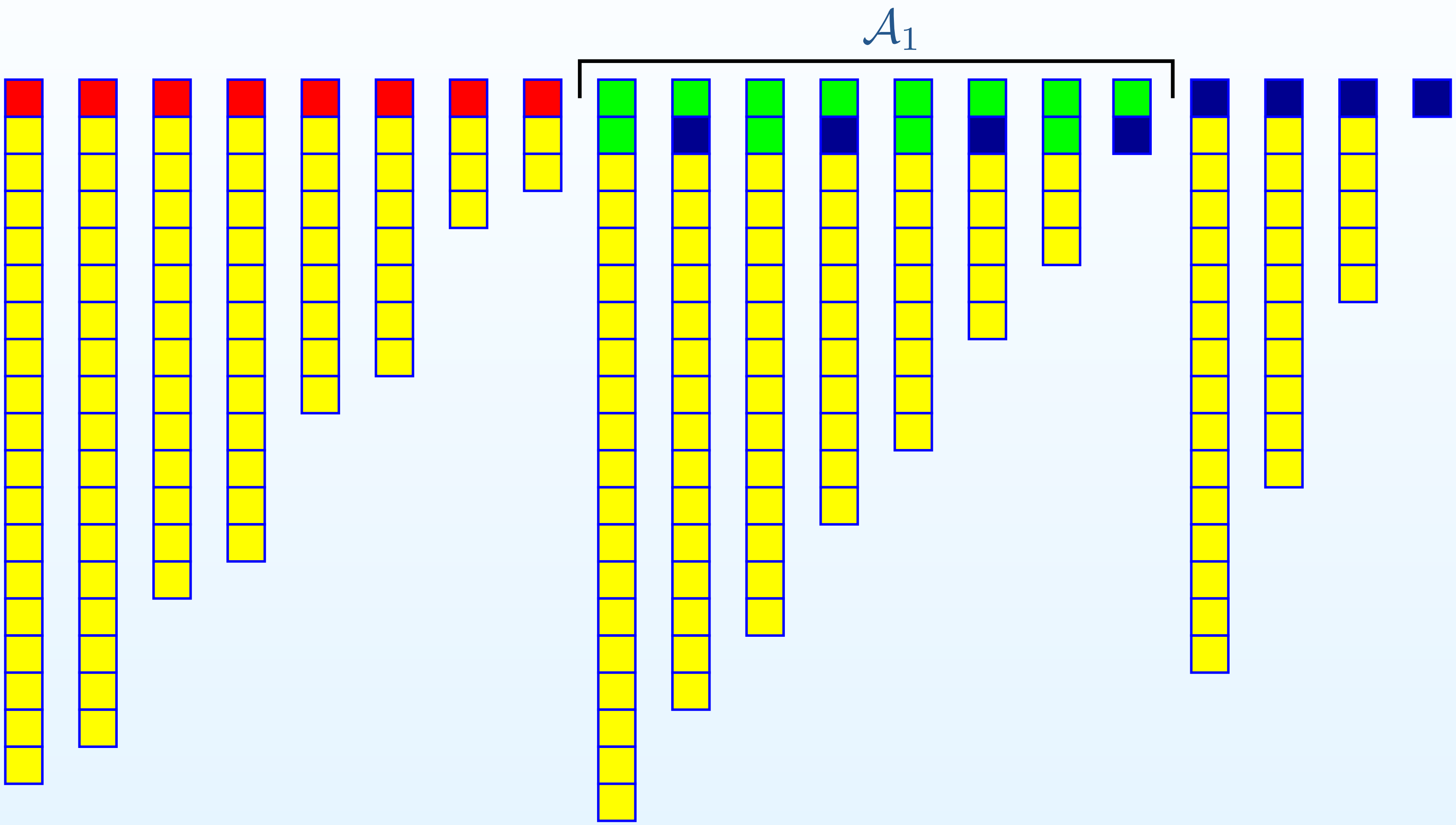
Suffix selection, first attempt



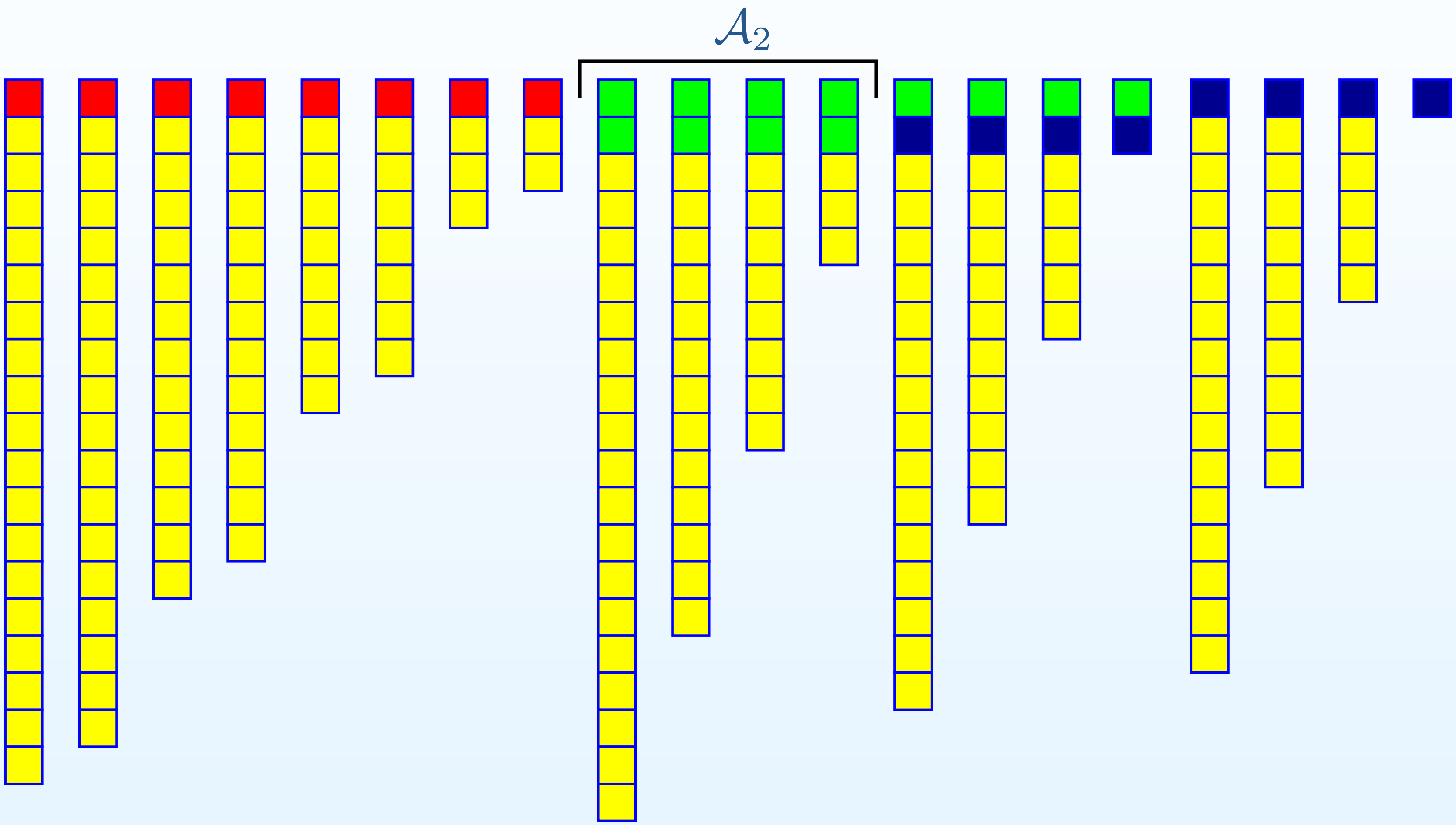
$k = 10$, $l_1 = 8$, Phase 1

Suffix selection, first attempt



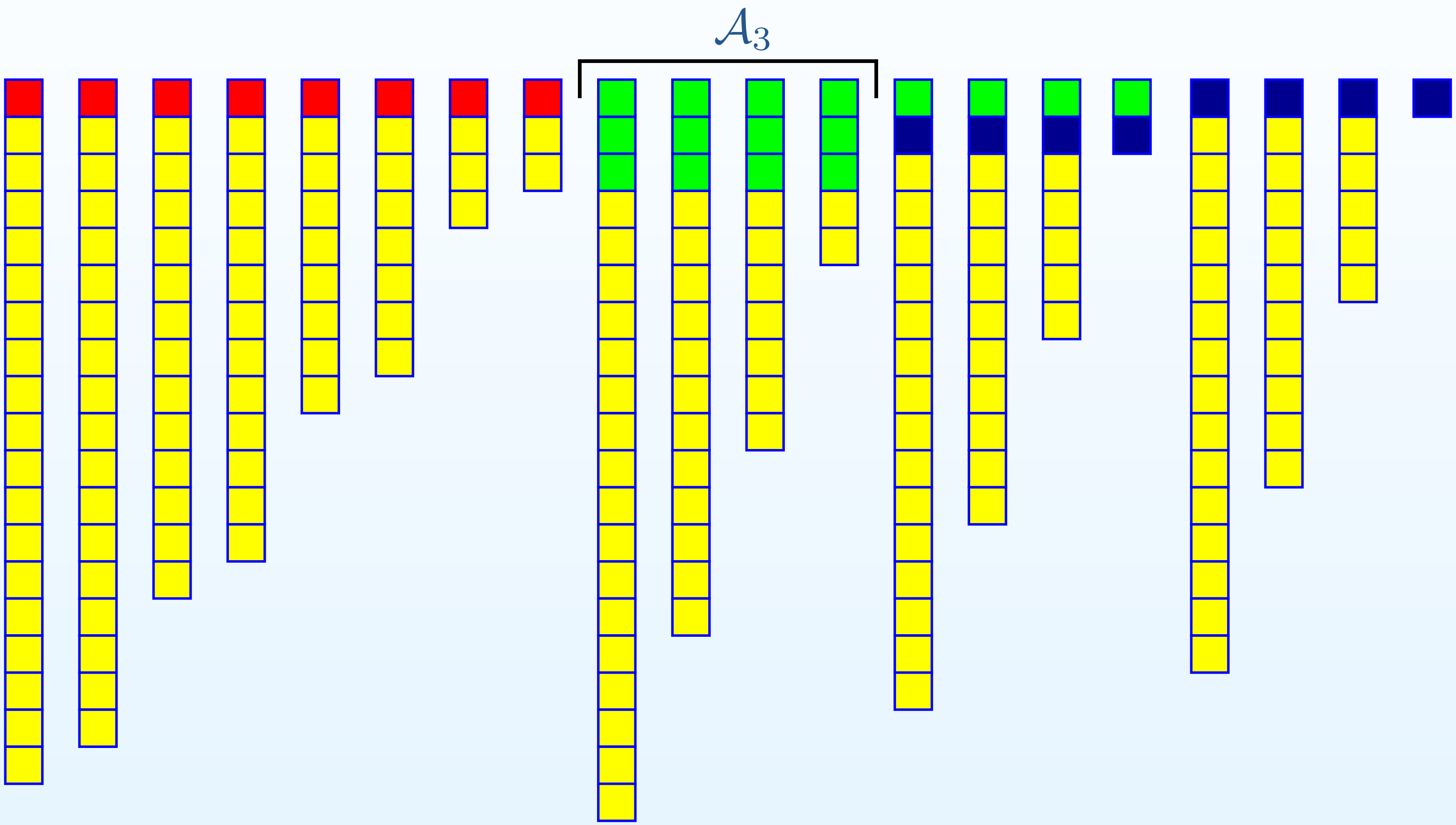
$k = 10$, $l_1 = 8$, Phase 1

Suffix selection, first attempt



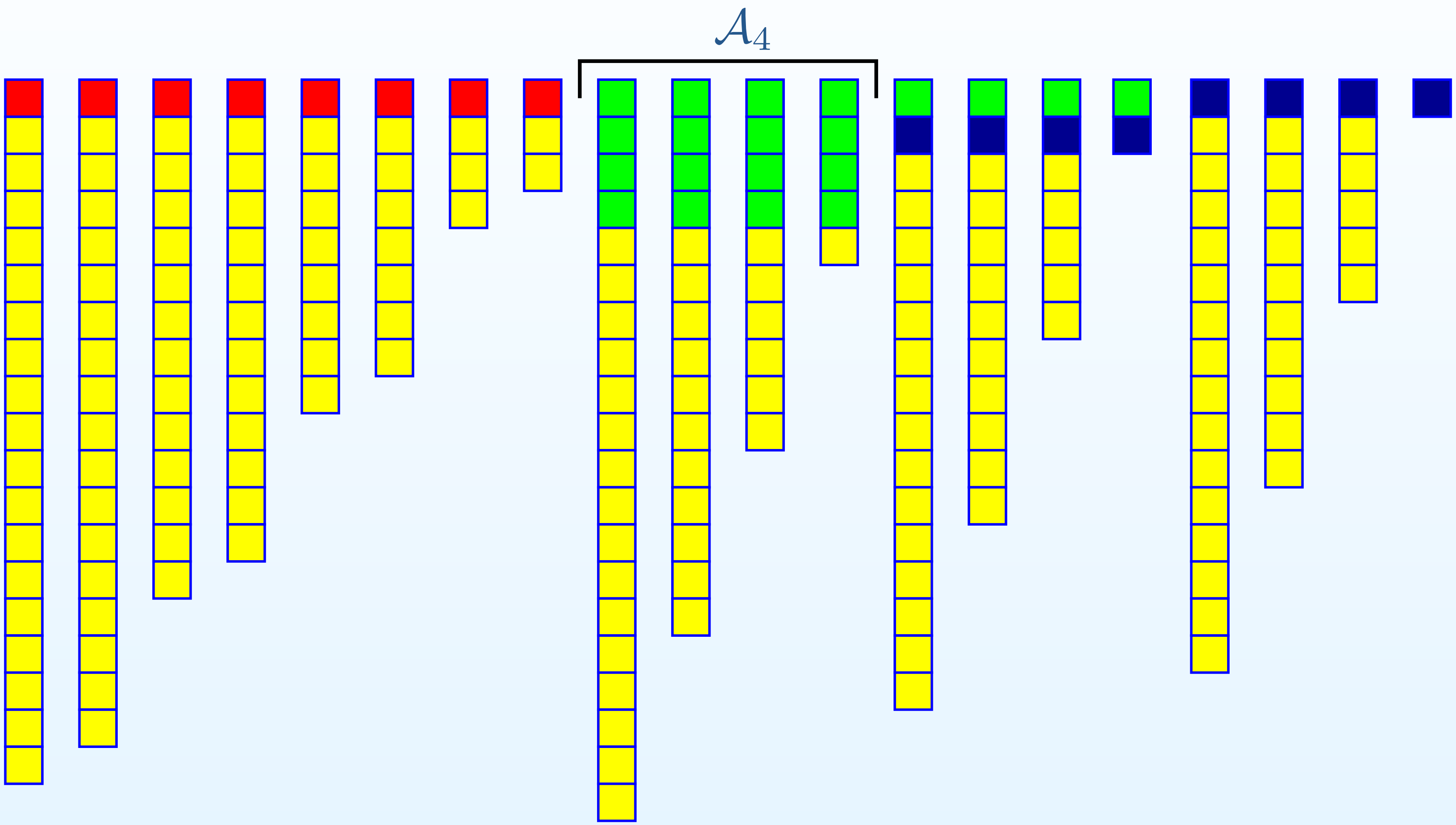
$k = 10$, $l_2 = 8$, Phase 2

Suffix selection, first attempt



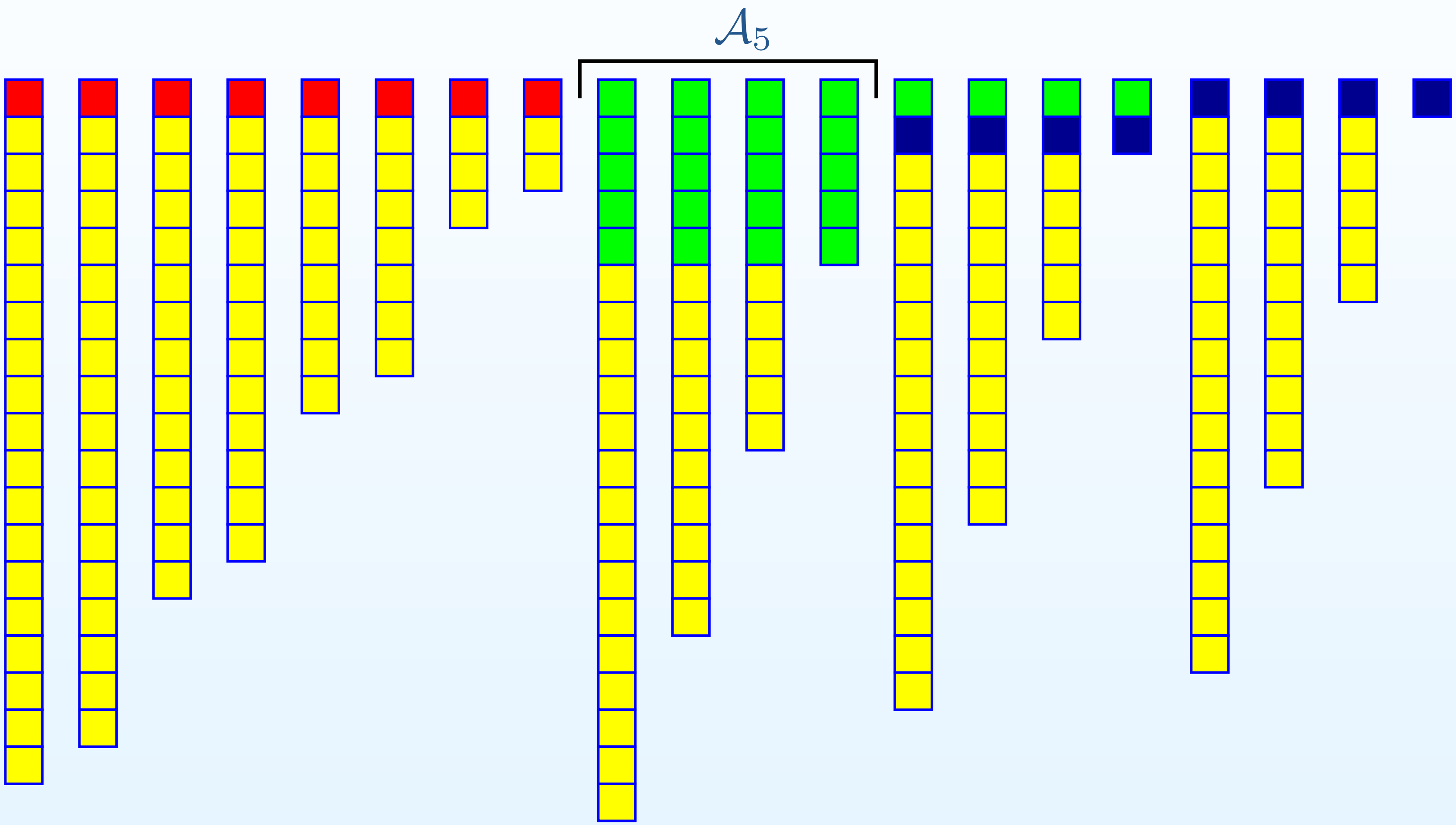
$k = 10$, $l_3 = 8$, Phase 3

Suffix selection, first attempt



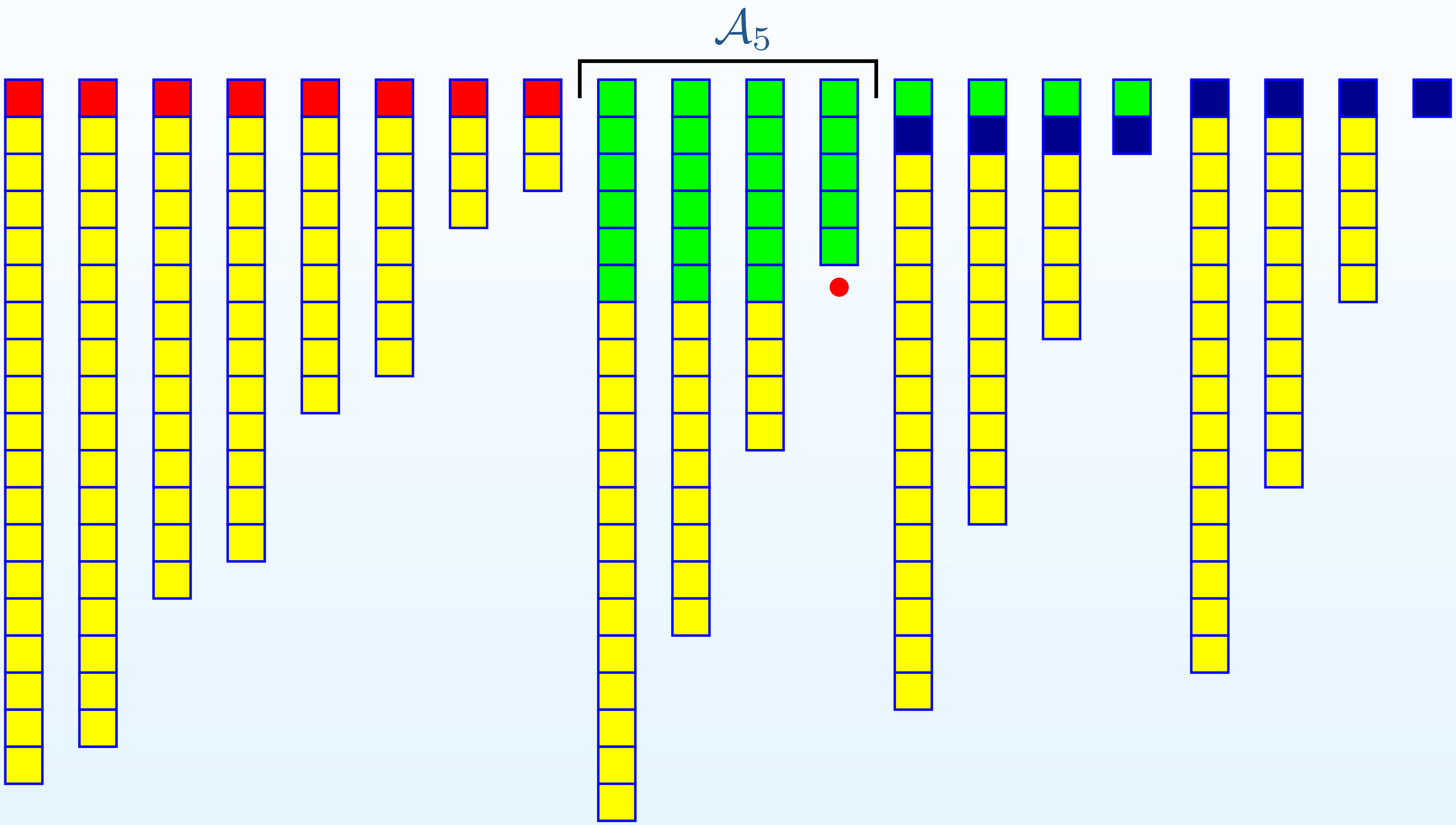
$k = 10$, $l_4 = 8$, Phase 4

Suffix selection, first attempt



$k = 10$, $l_5 = 8$, Phase 5

Suffix selection, first attempt



$k = 10$, $l_5 = 8$, Phase 5

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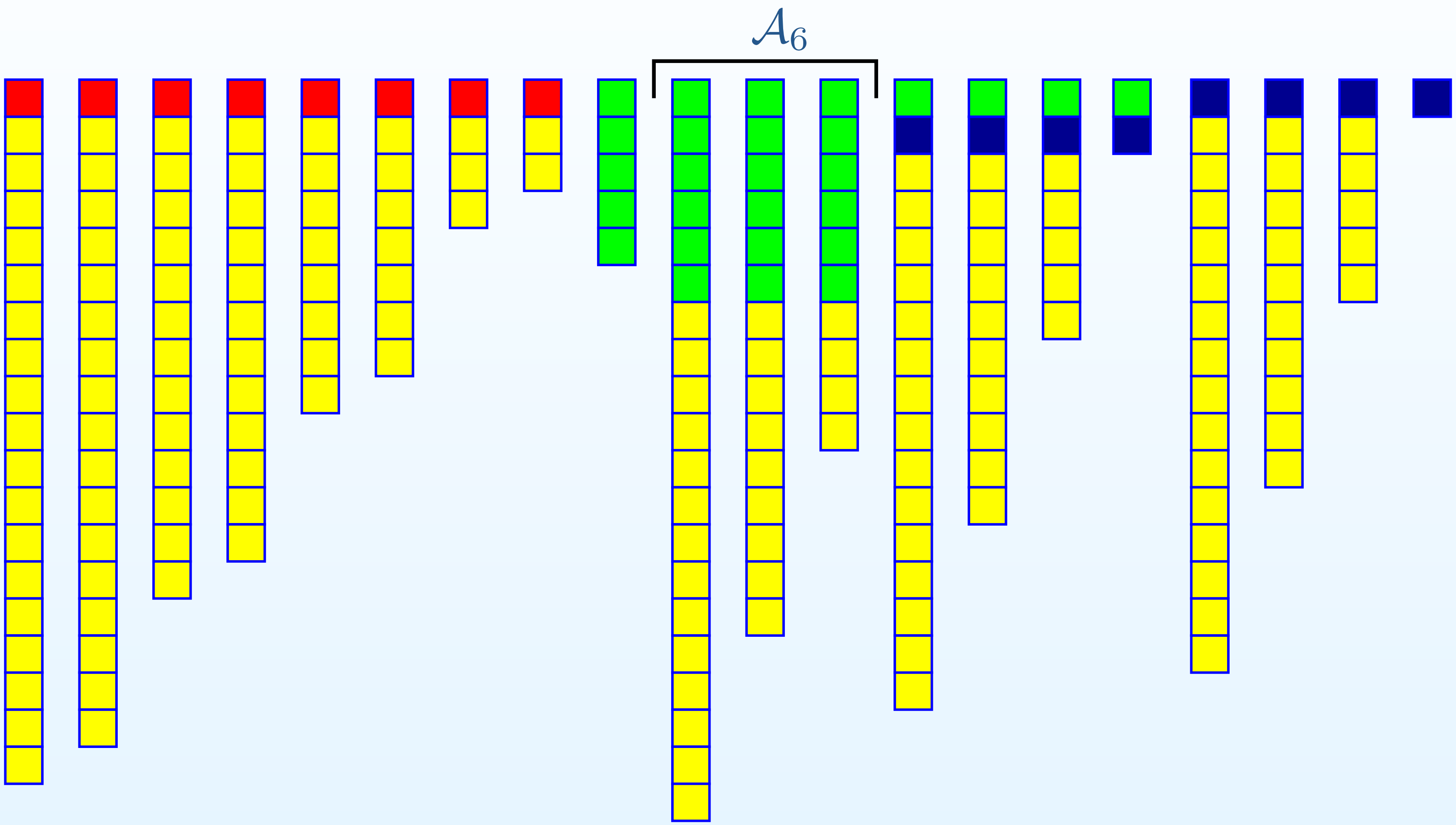

$k = 10$, $l_6 = 9$, Phase 6

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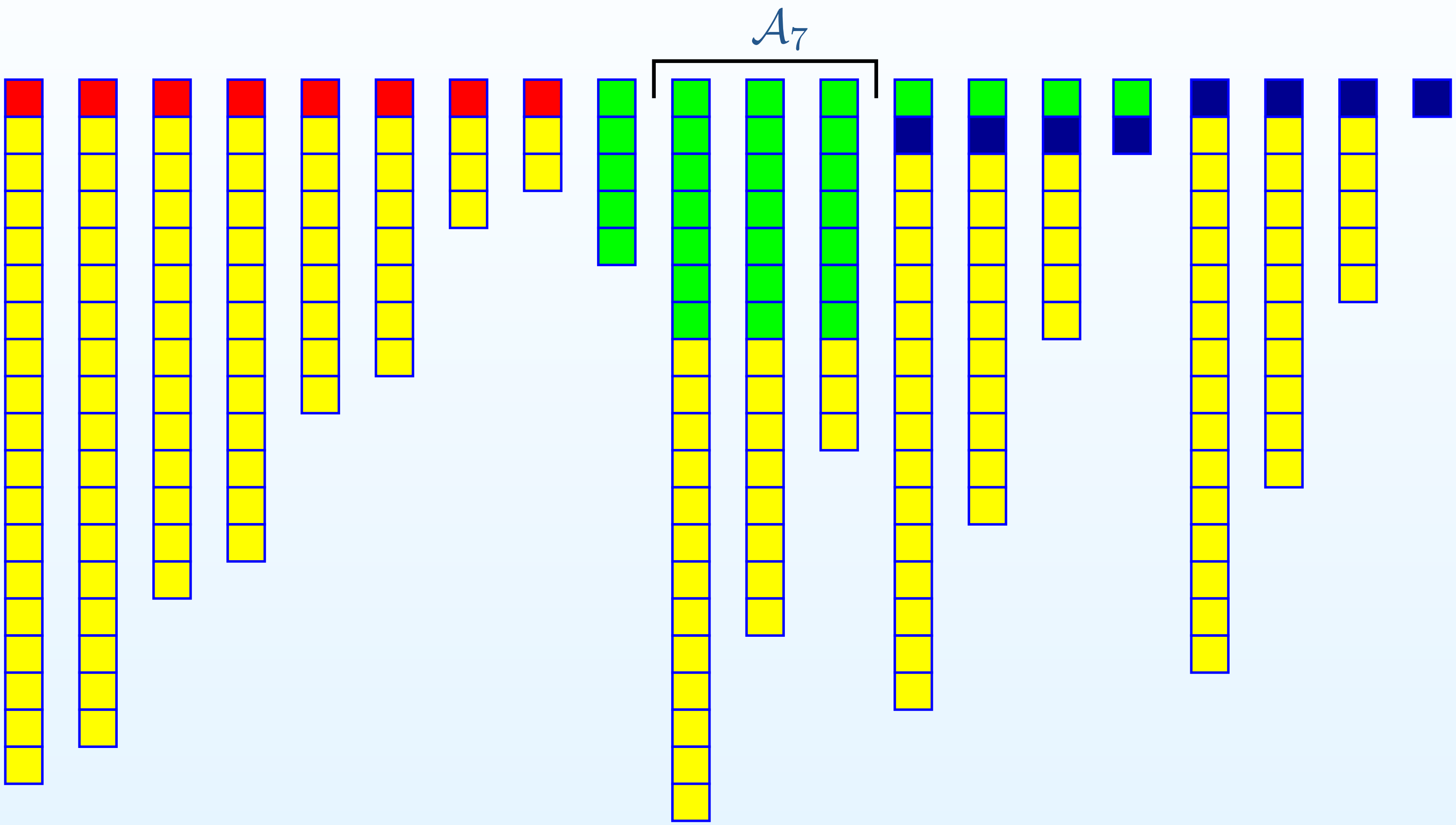
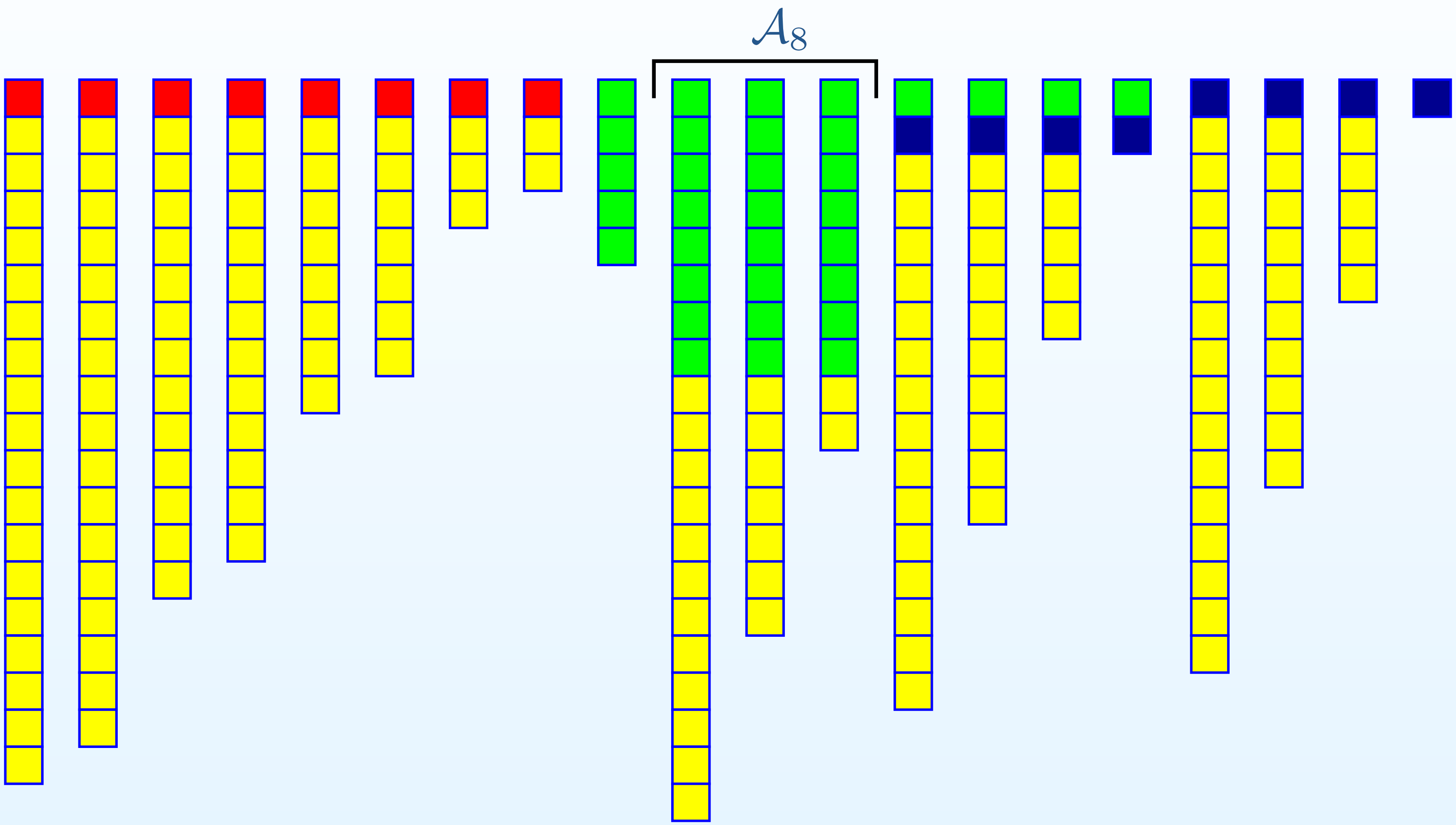


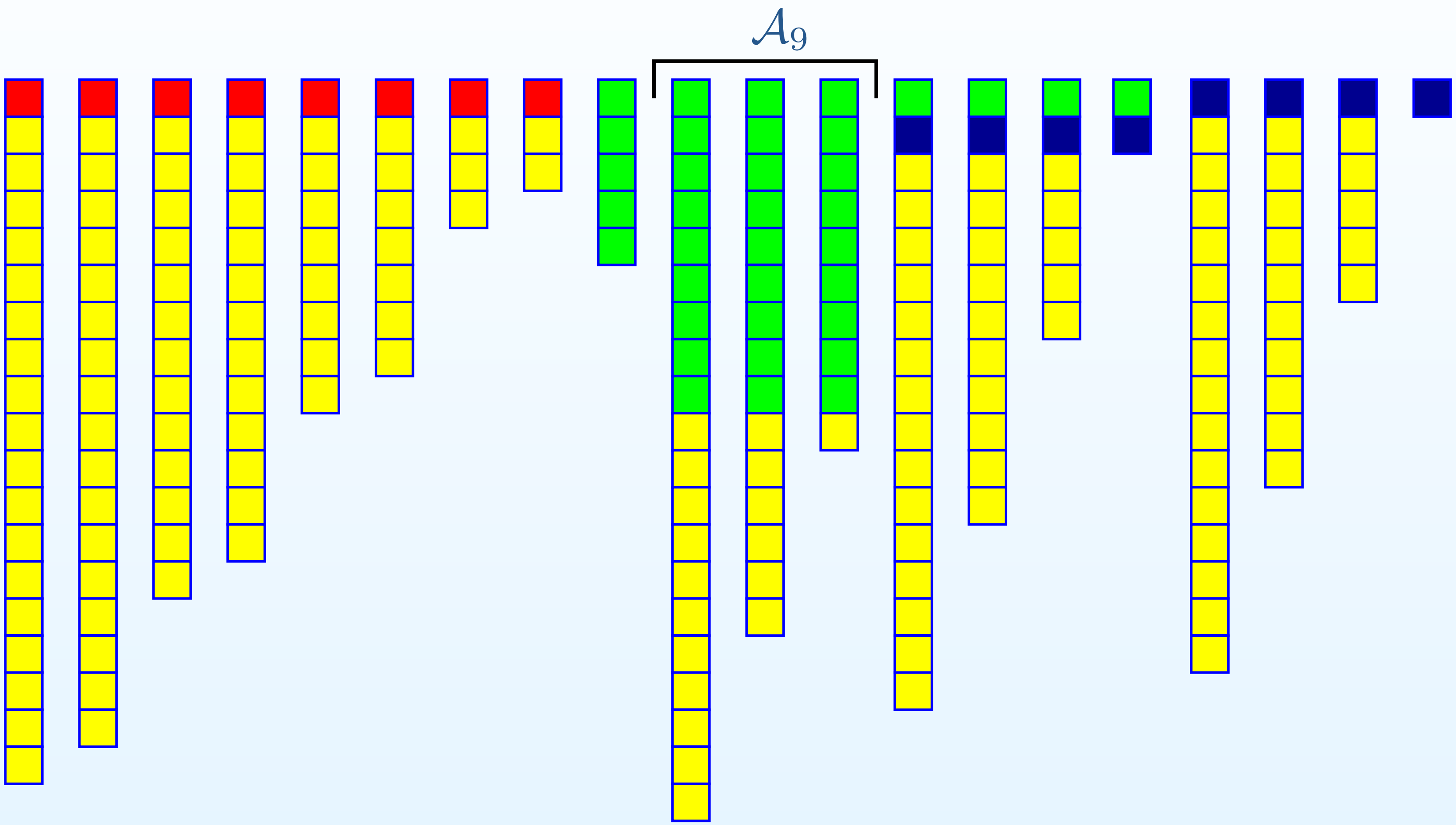
$k = 10$, $l_7 = 9$, Phase 7

Suffix selection, first attempt



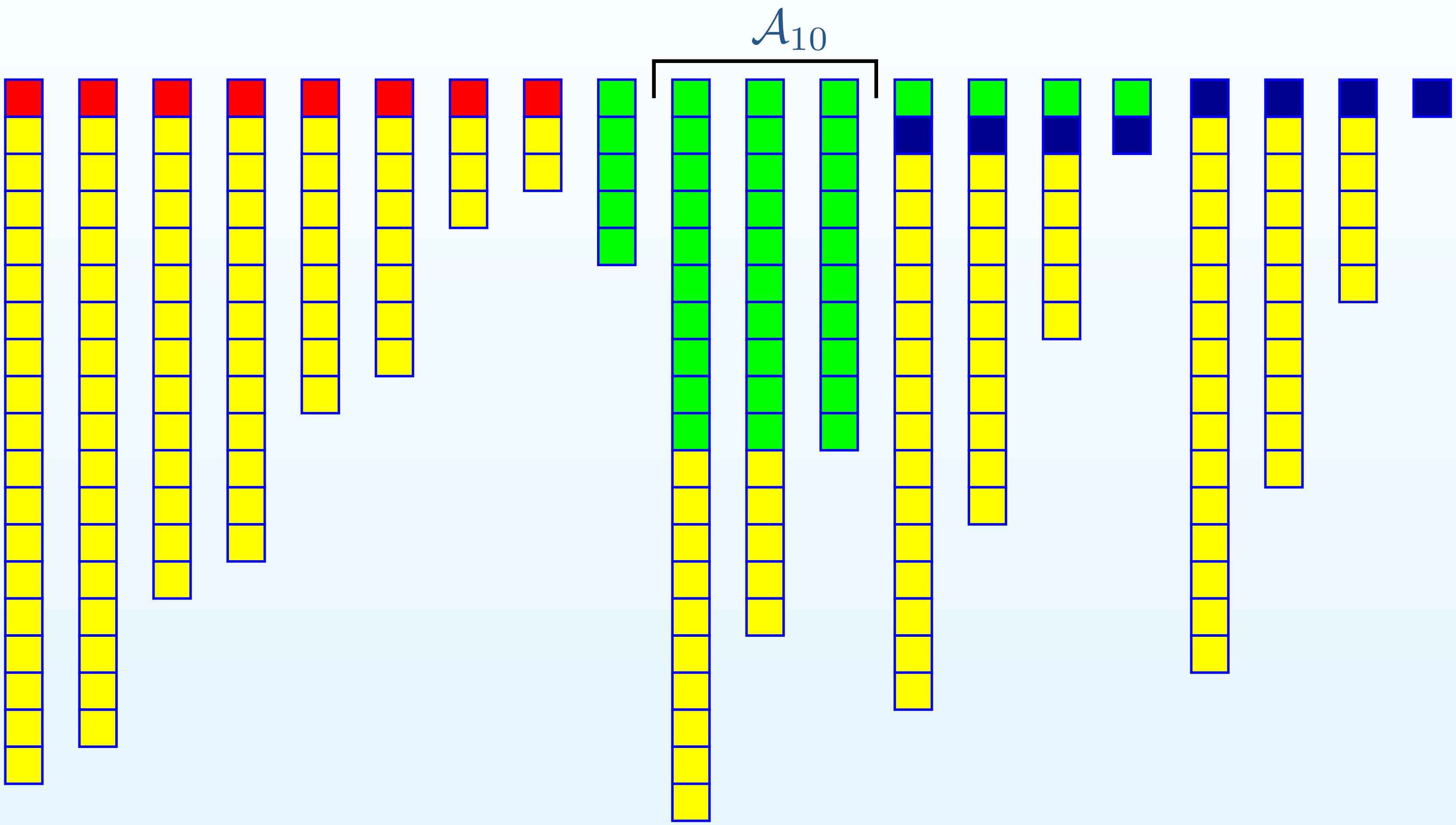
$k = 10$, $l_8 = 9$, Phase 8

Suffix selection, first attempt



$k = 10$, $l_9 = 9$, Phase 9

Suffix selection, first attempt



$k = 10$, $l_{10} = 9$, Phase 10

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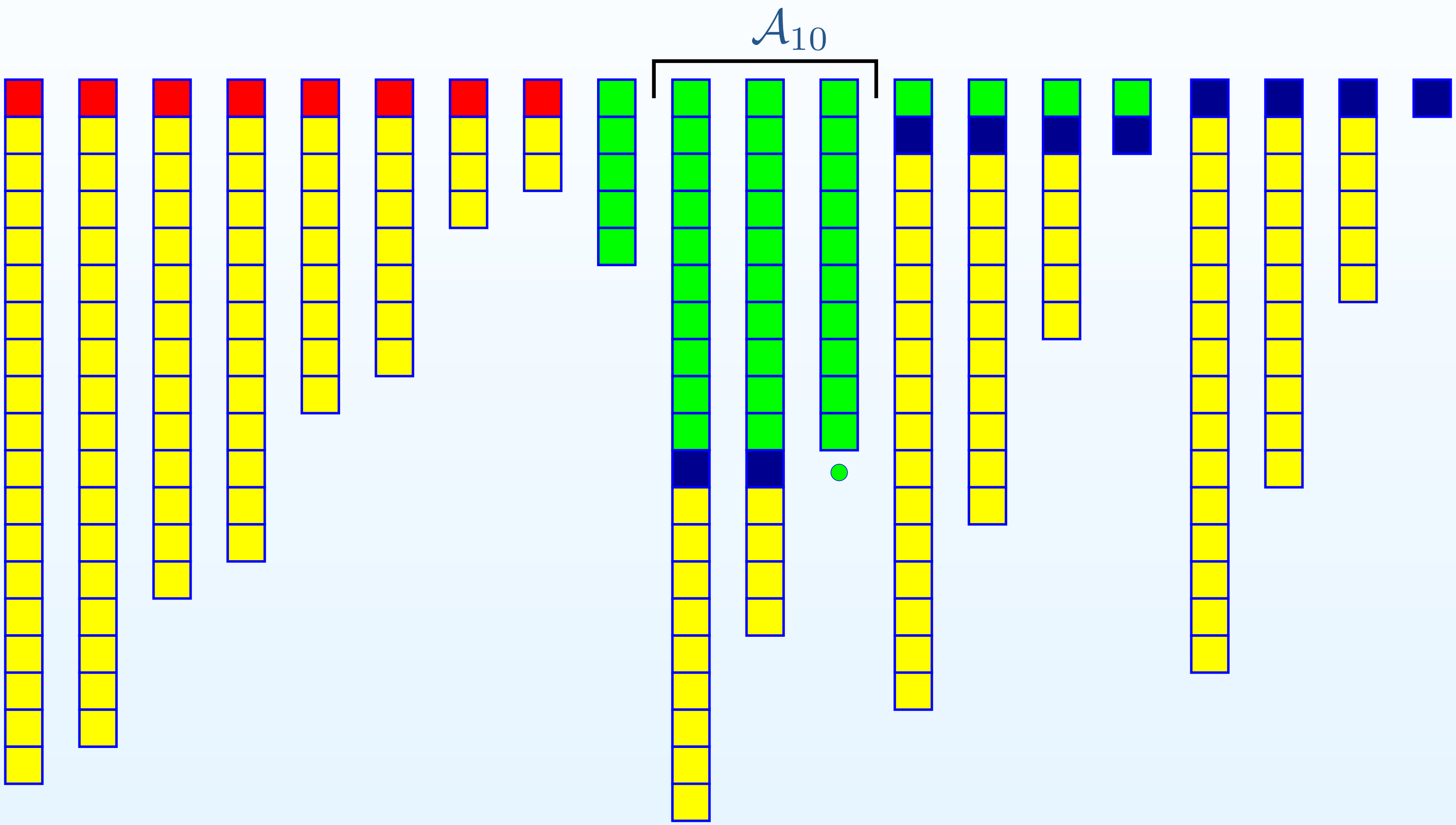

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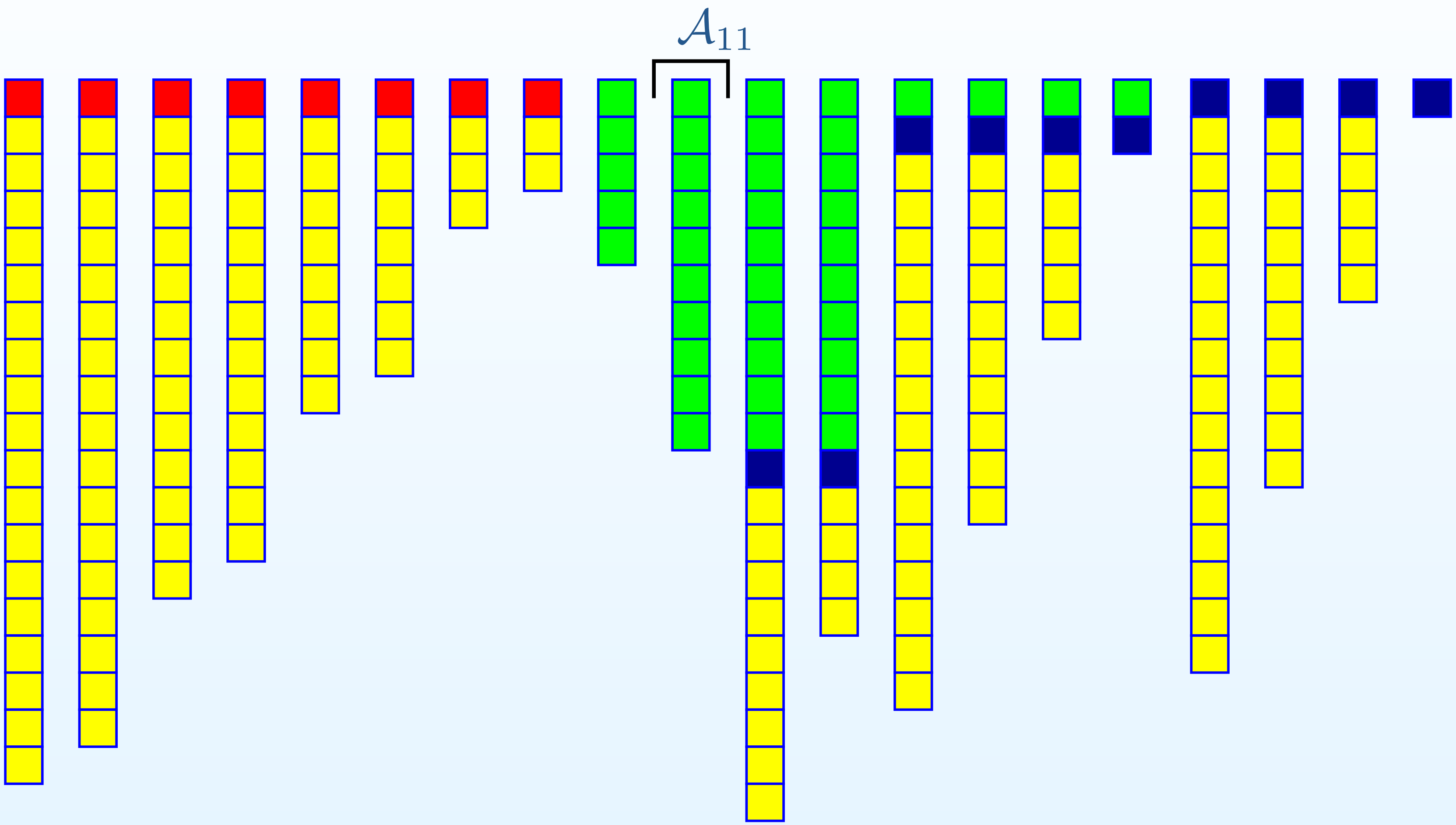

$k = 10$, $l_{11} = 9$, Phase 11

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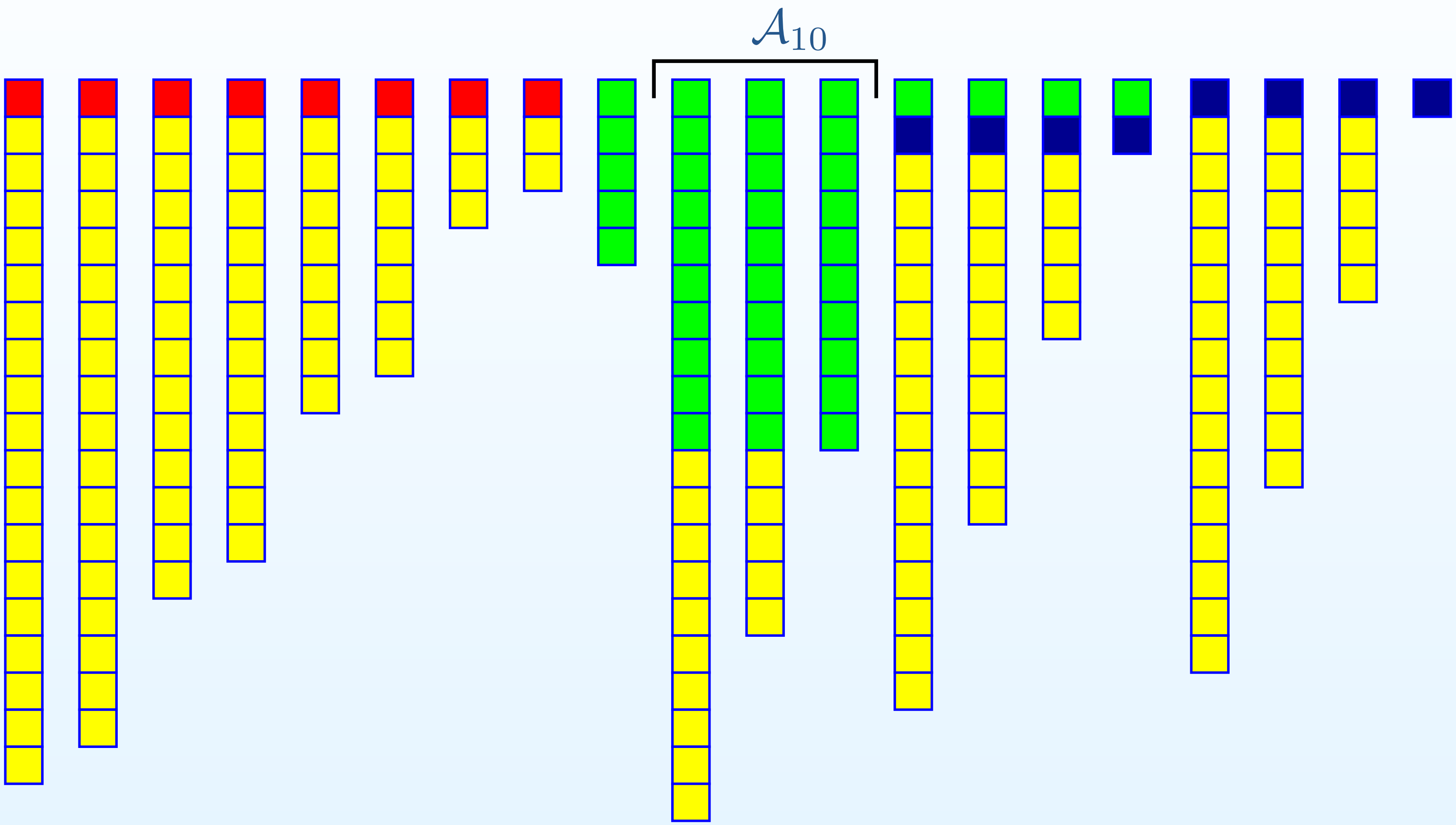
  - *By reusing the work done on inactive suffixes.*

Suffix selection: collisions of active suffixes



$k = 10$, $l_{10} = 9$, Phase 10

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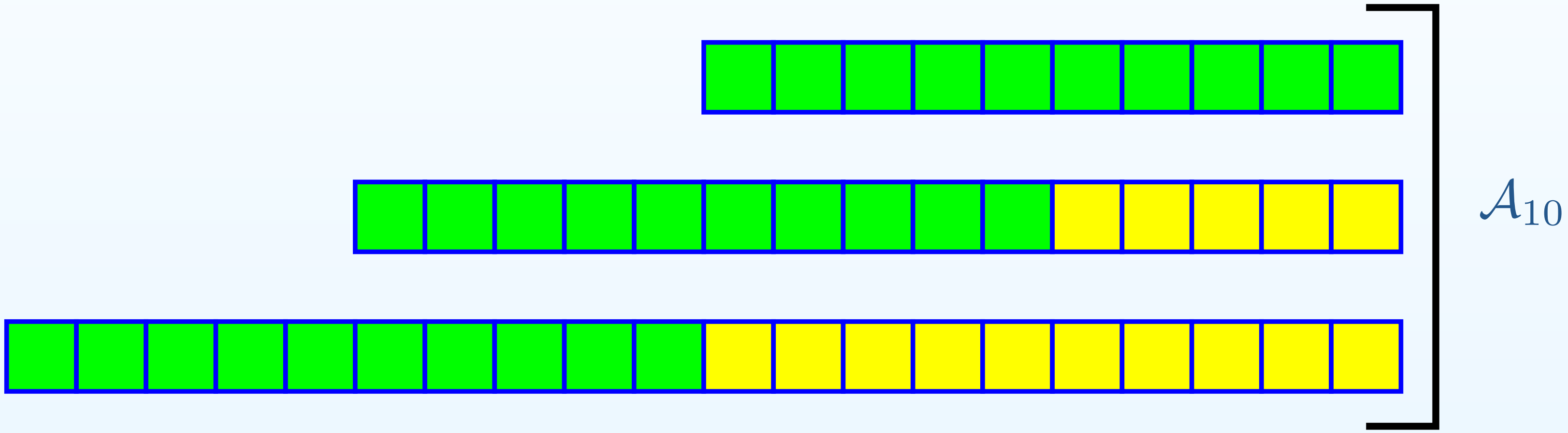

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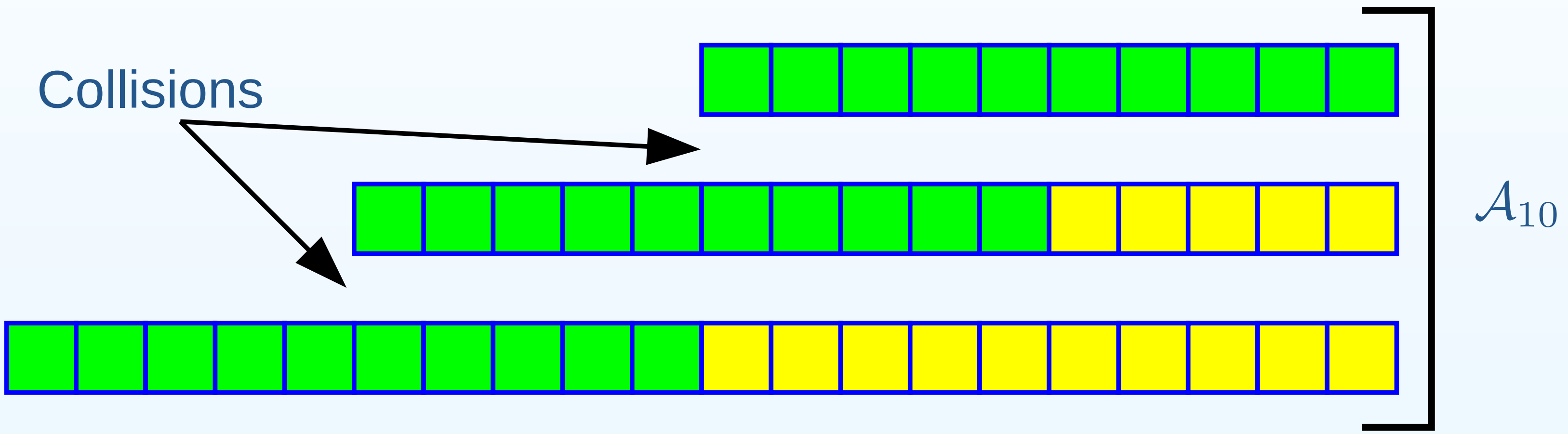

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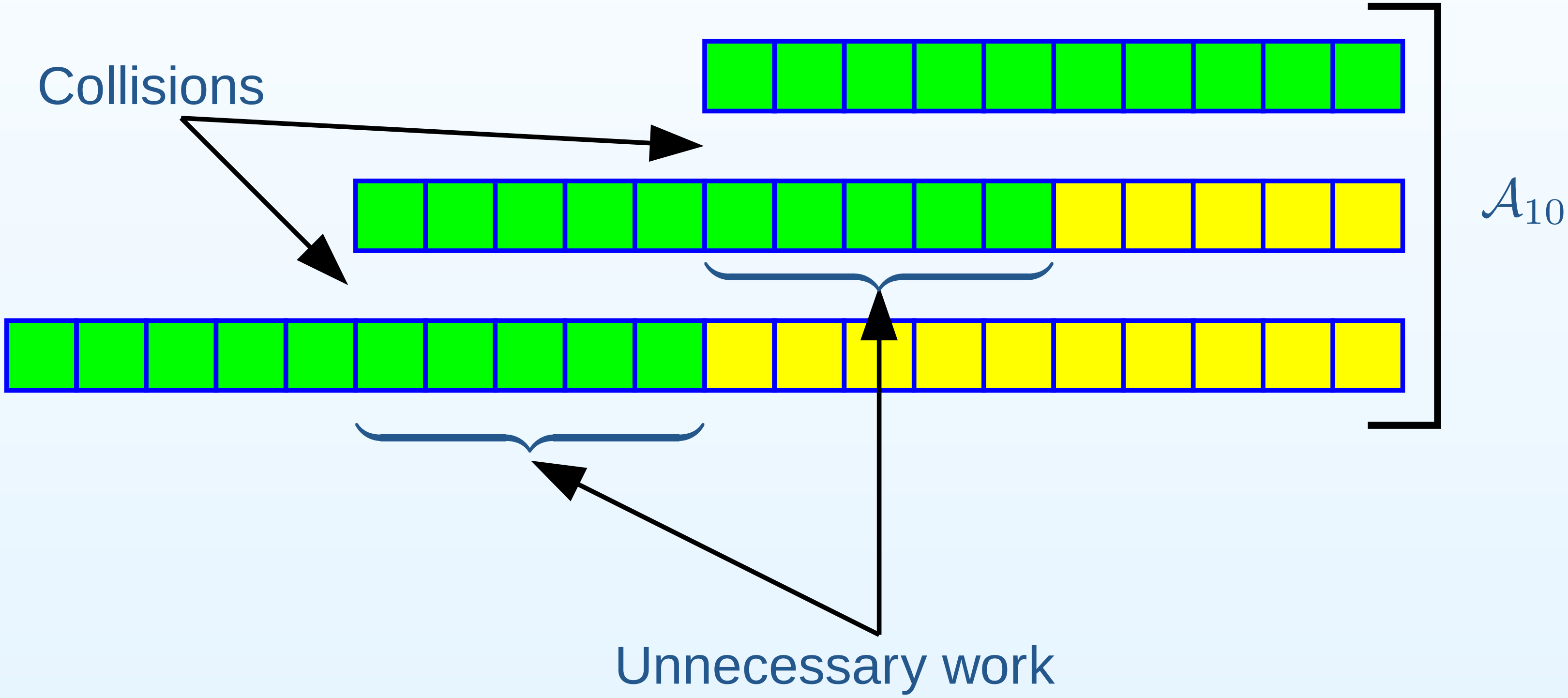

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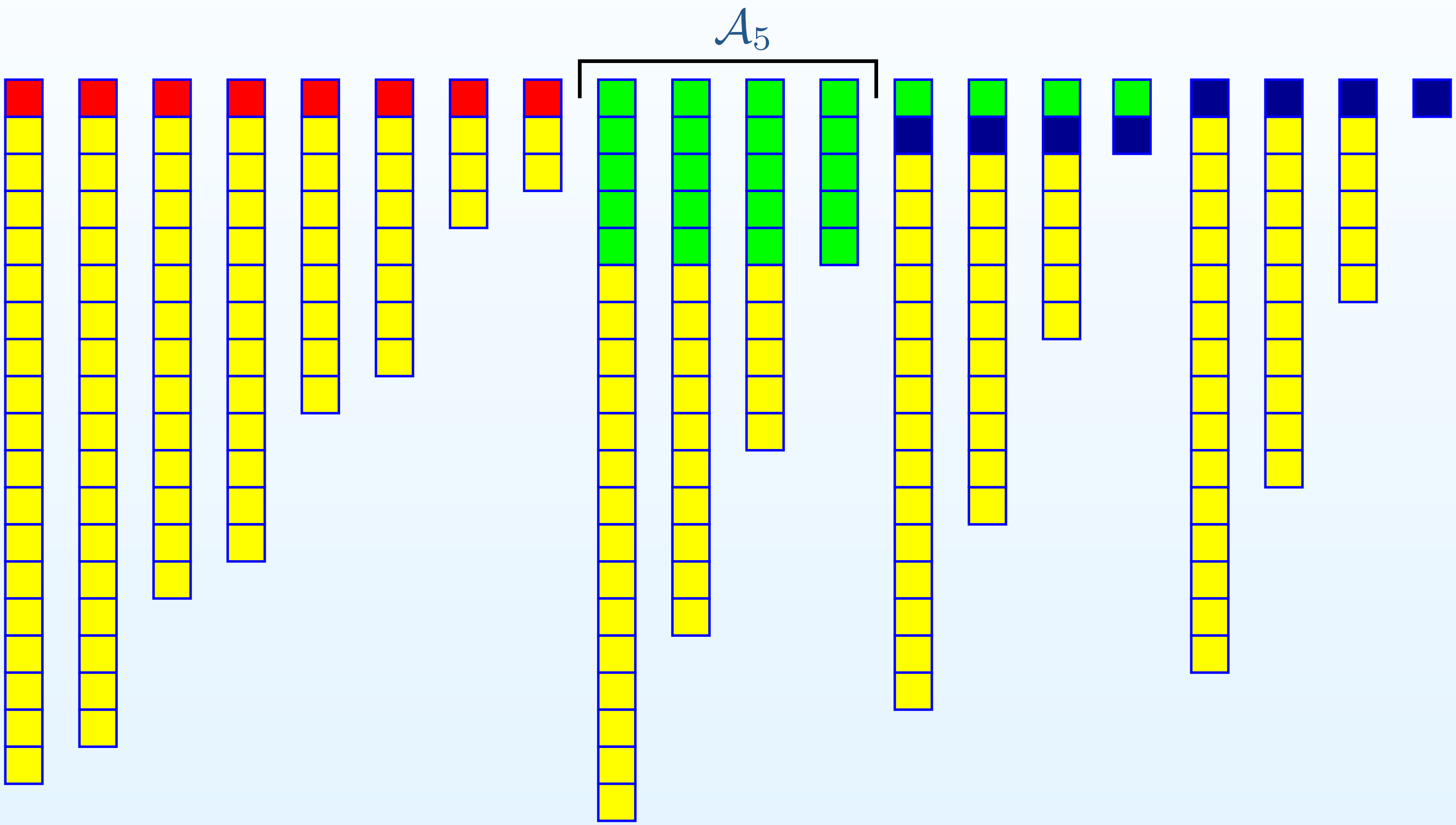
In the first attempt, with any phase transition

- we tried to enlarge *by just one element* the extent of each active suffix
- while completely *ignoring the emerging of collisions*.

Suffix selection, second attempt: exploiting collisions



$k = 10$, $l_5 = 8$, Phase 5

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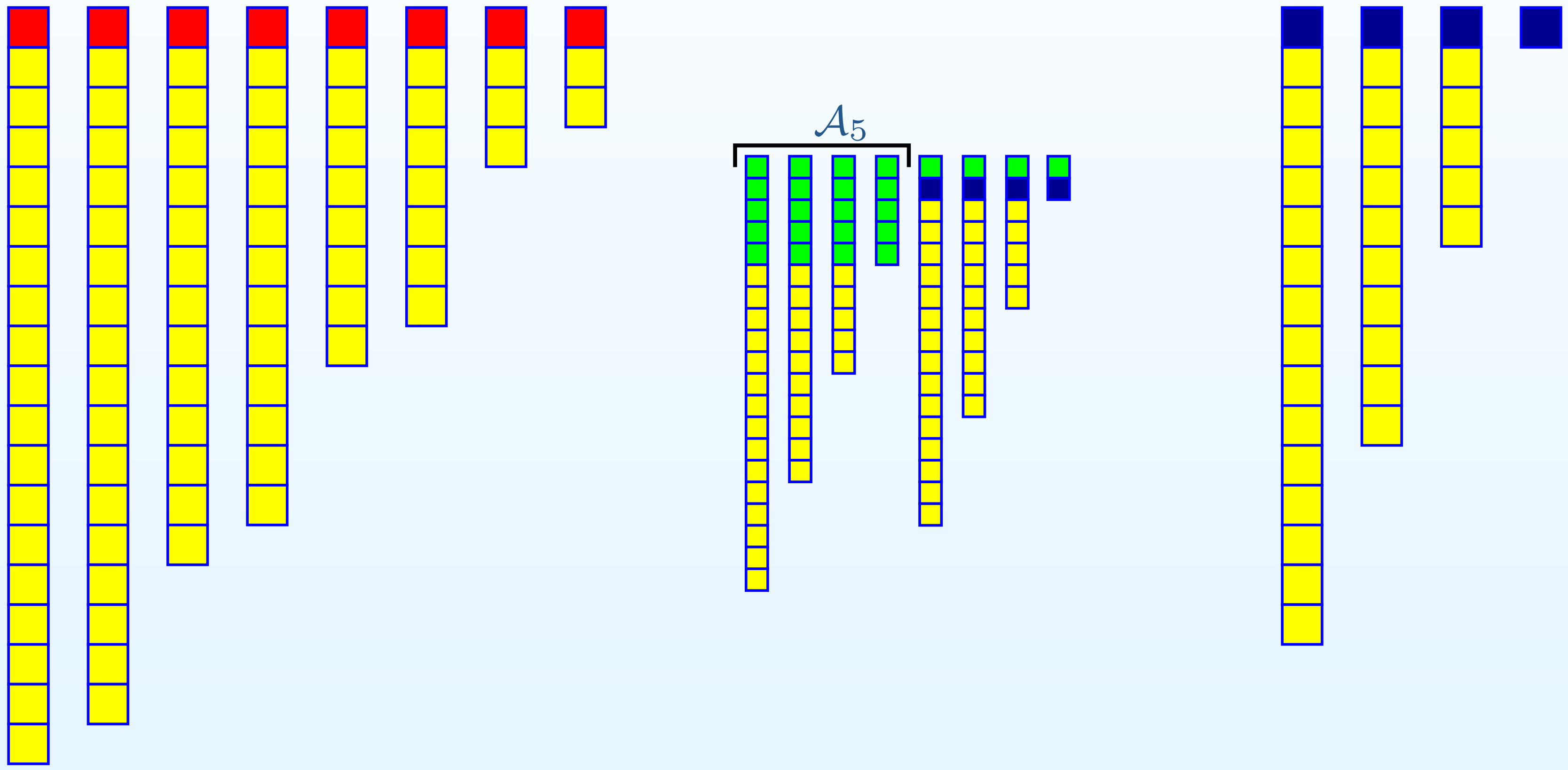

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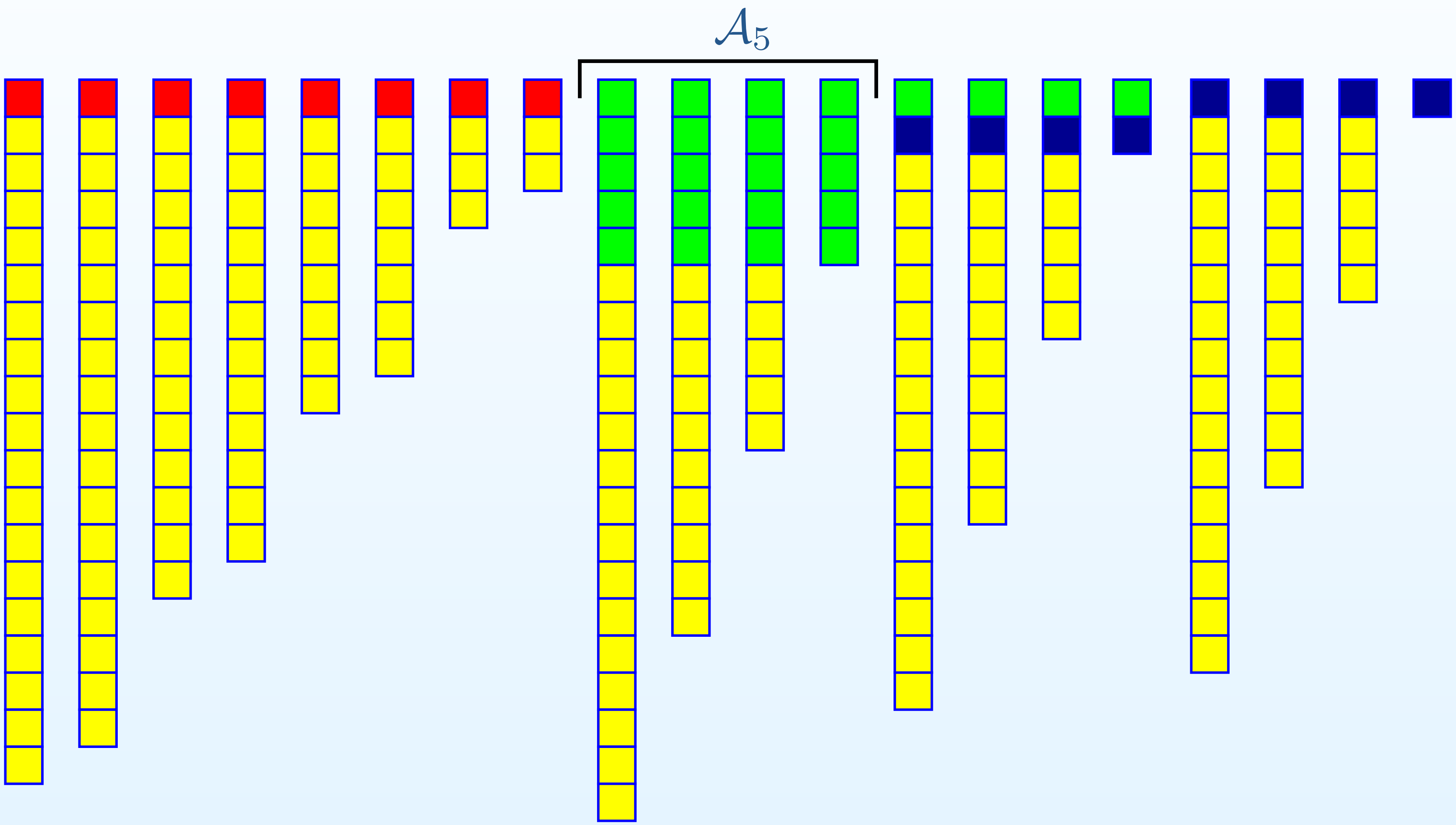

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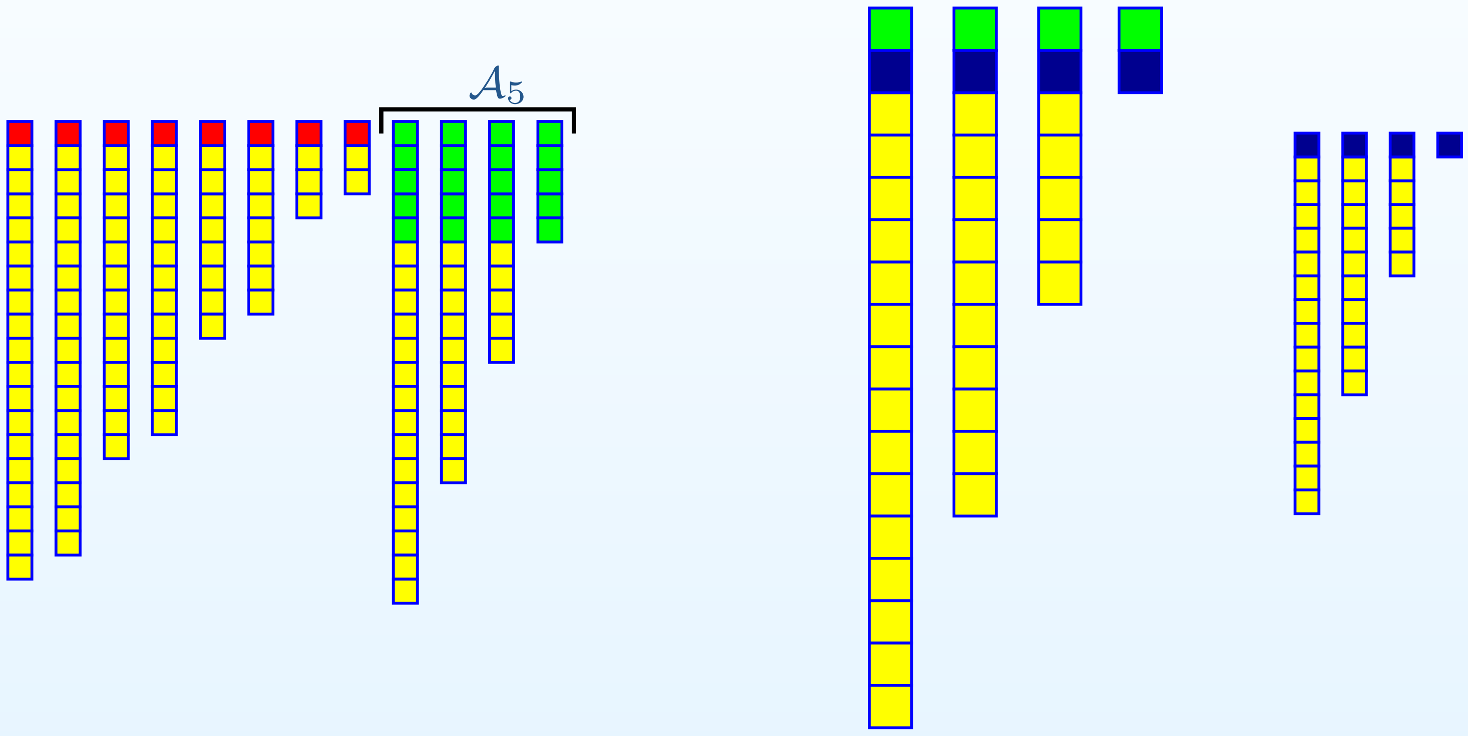

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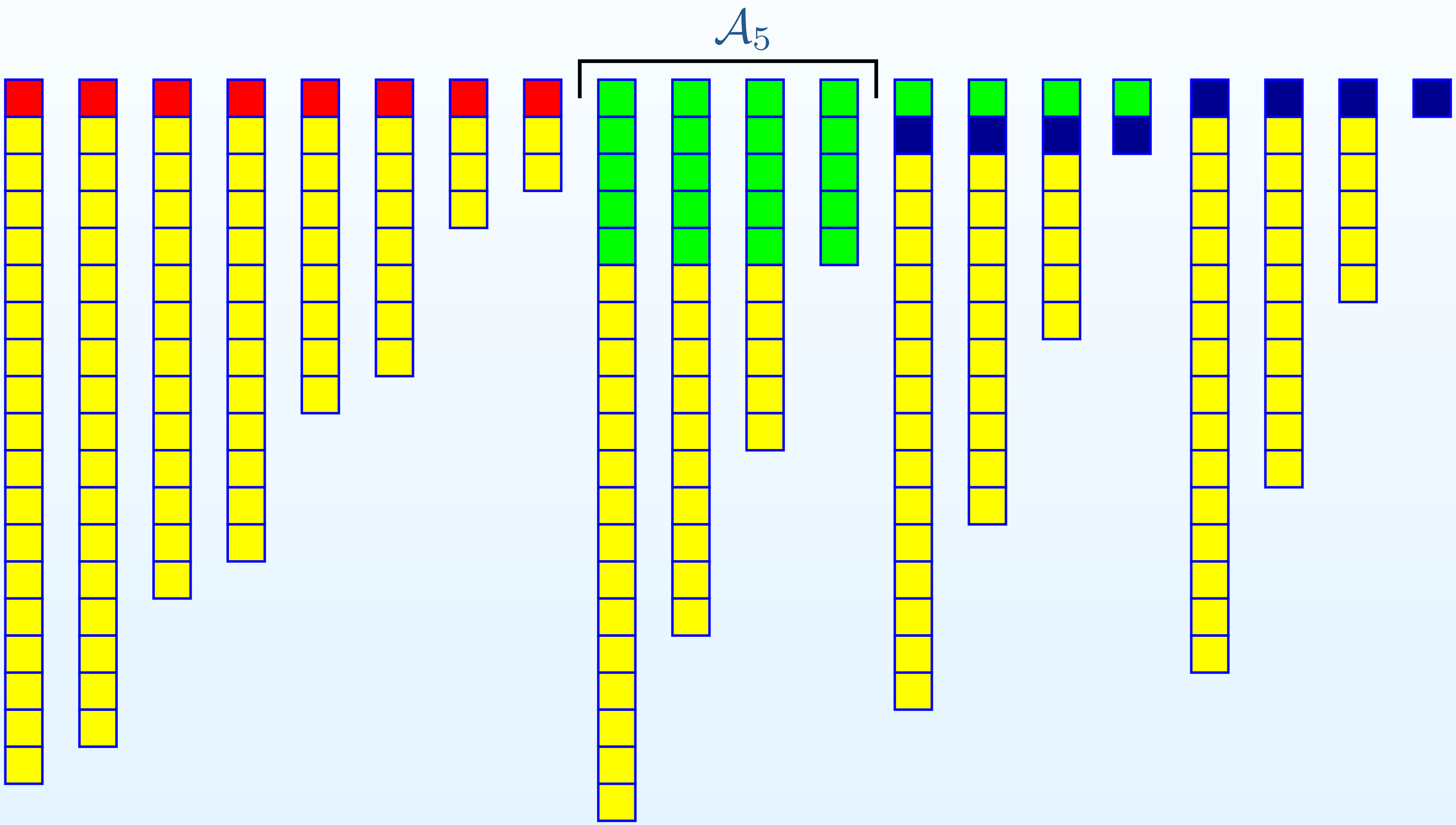

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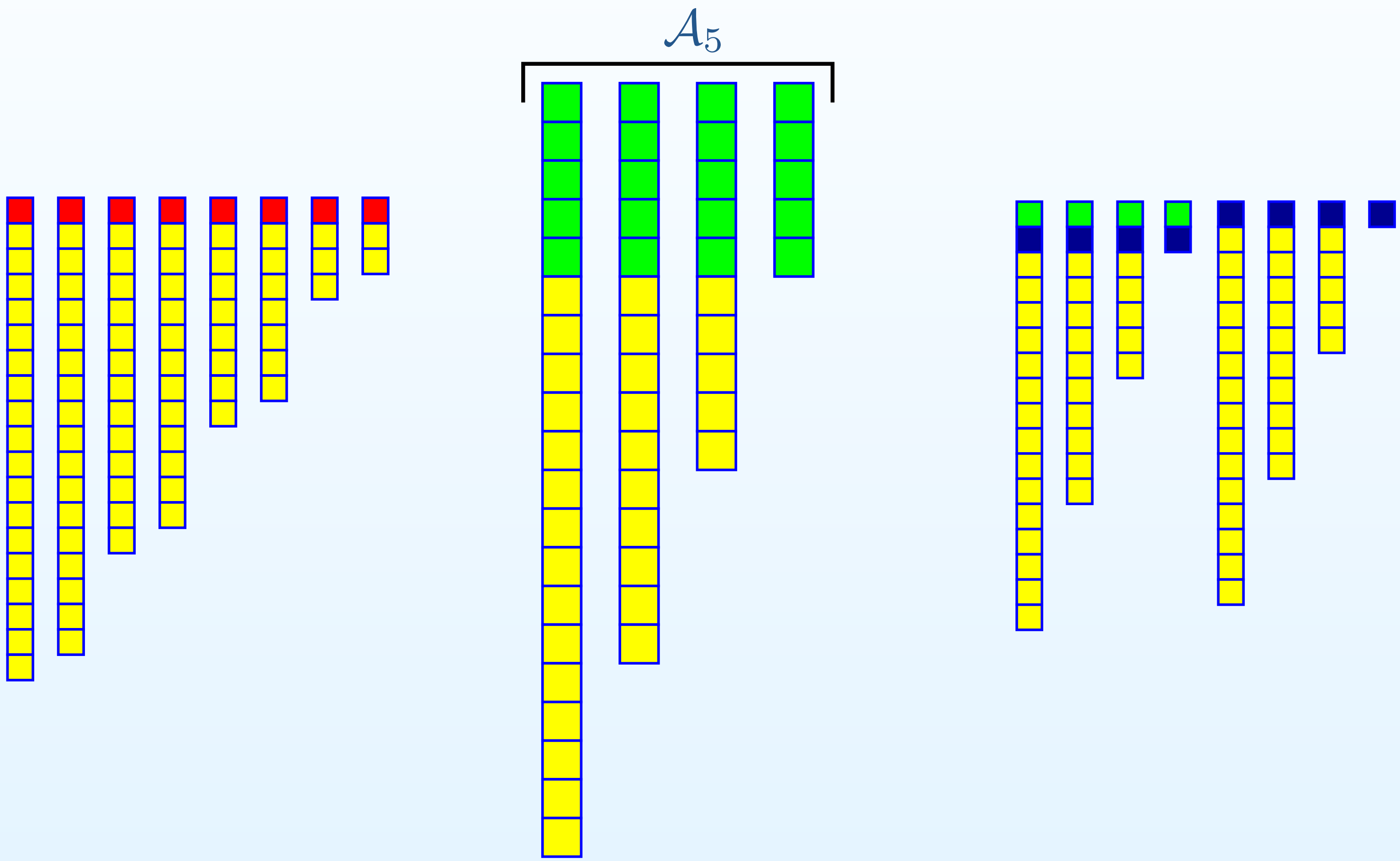

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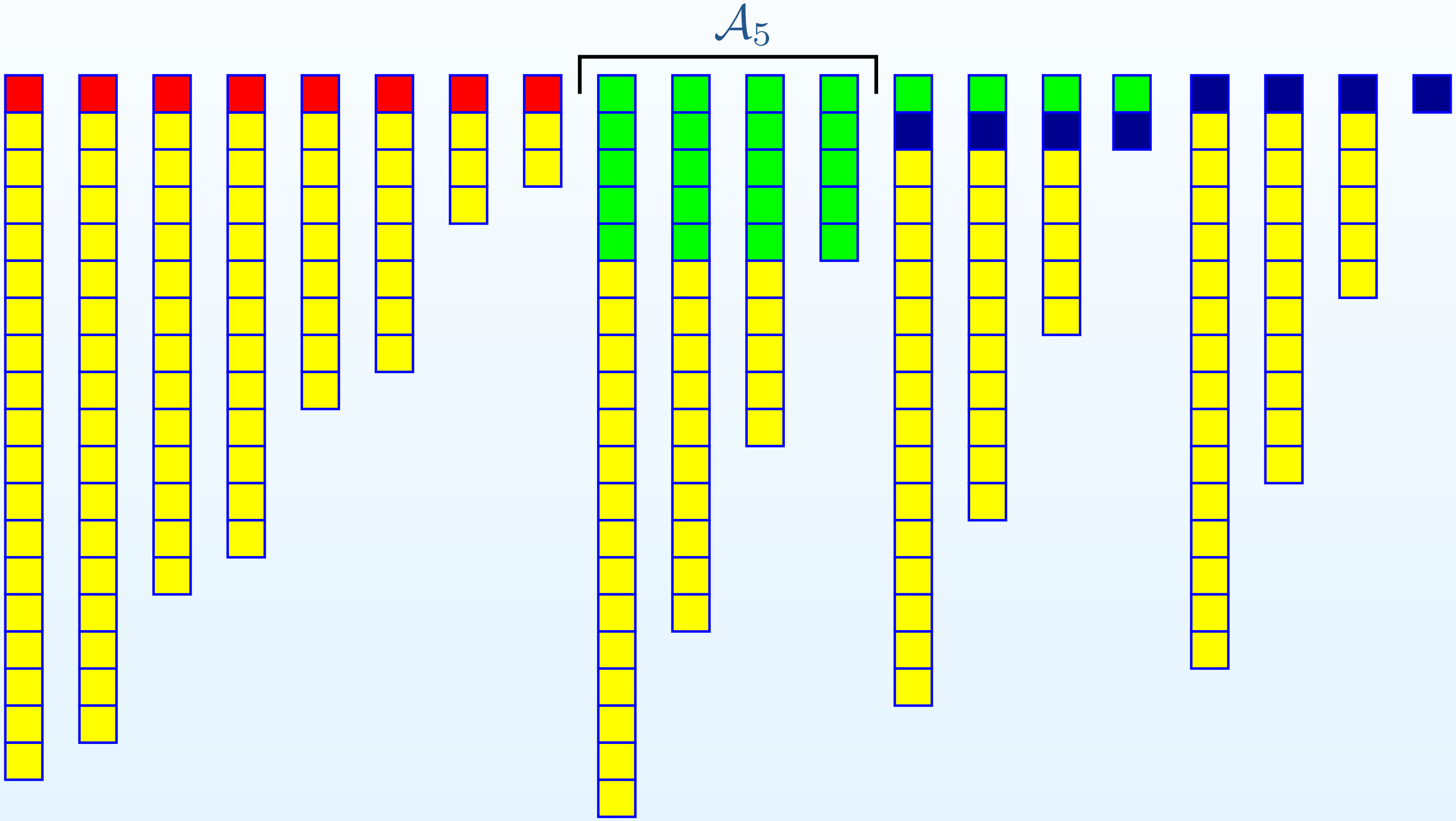

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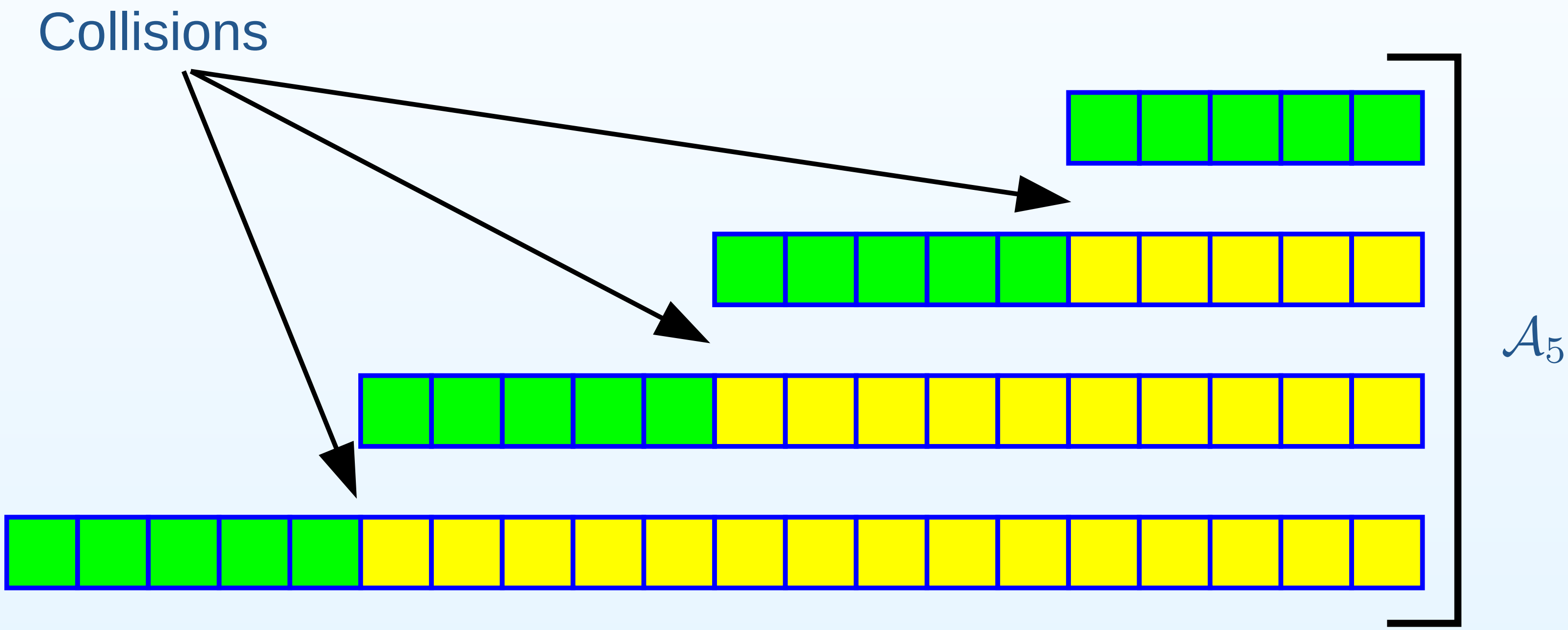

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  for an integer r_i.

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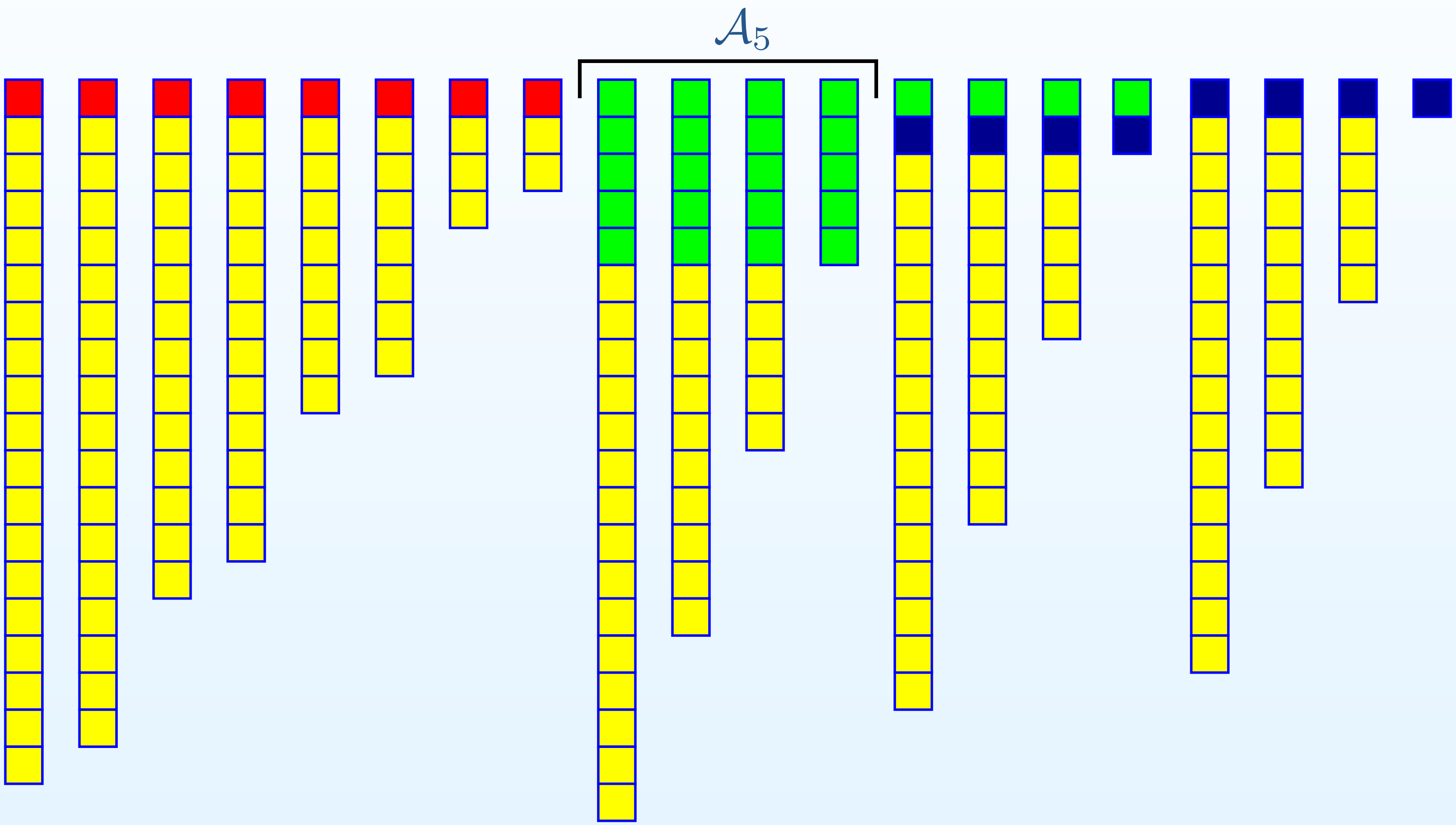
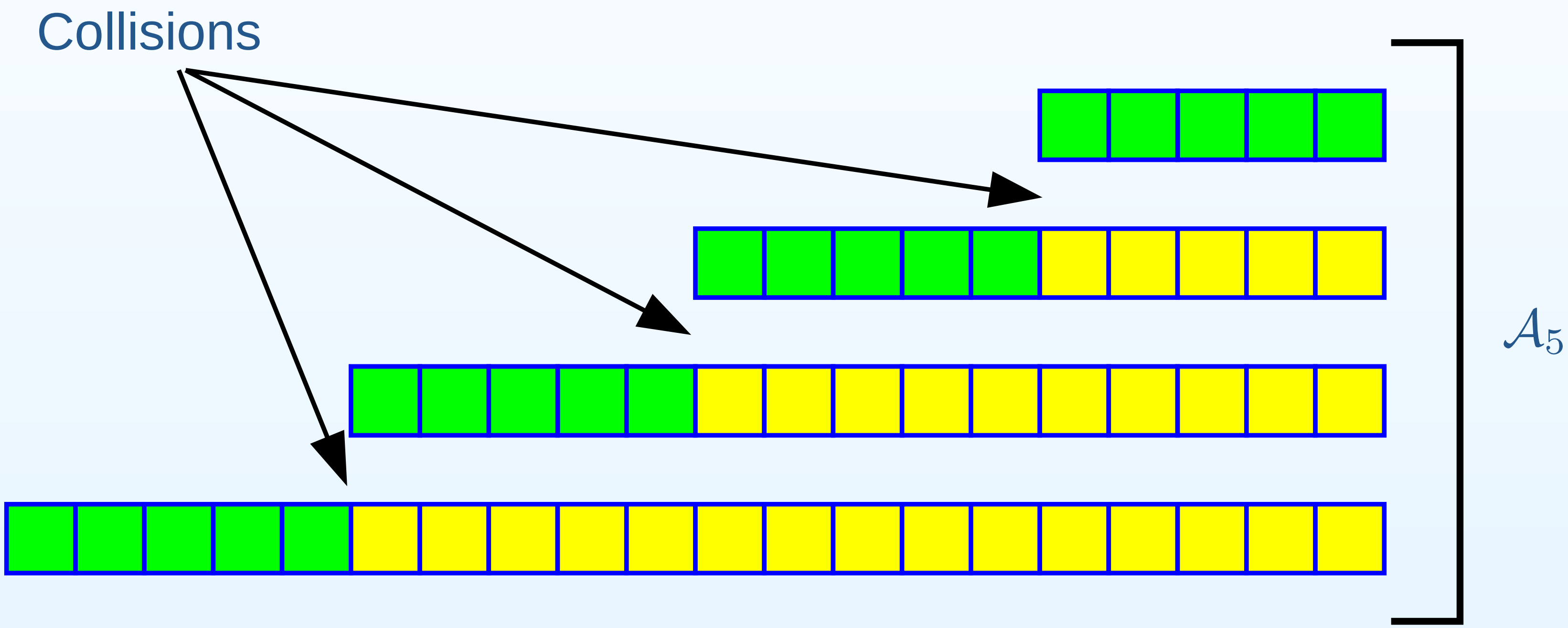
  for an integer r_i.
- Therefore, *the prospective extents* of any two active suffixes *can be compared in $O(1)$ time*.

Suffix selection, second attempt: exploiting collisions



$k = 10$, $l_5 = 8$, Phase 5

Suffix selection, second attempt: exploiting collisions



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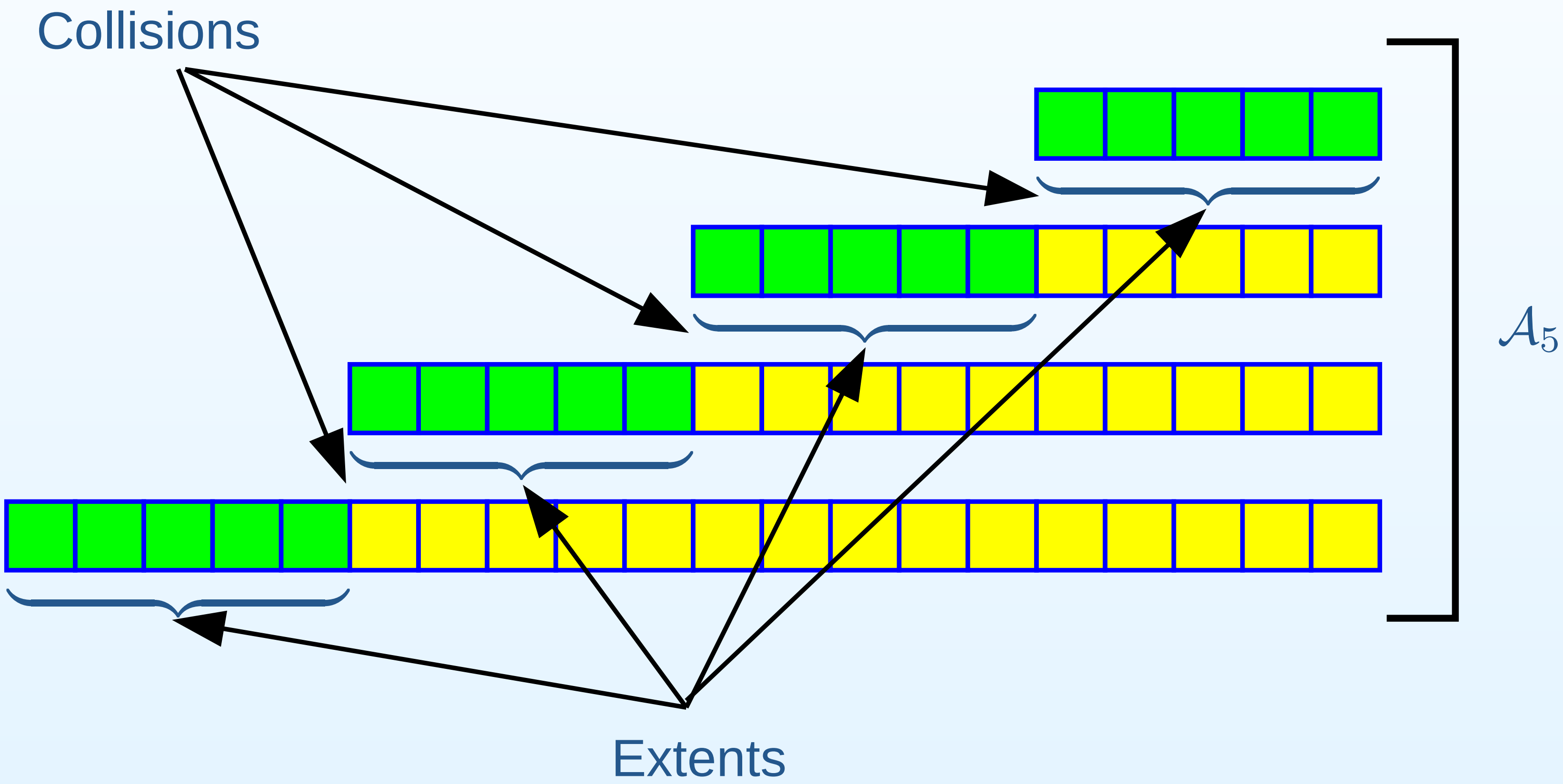

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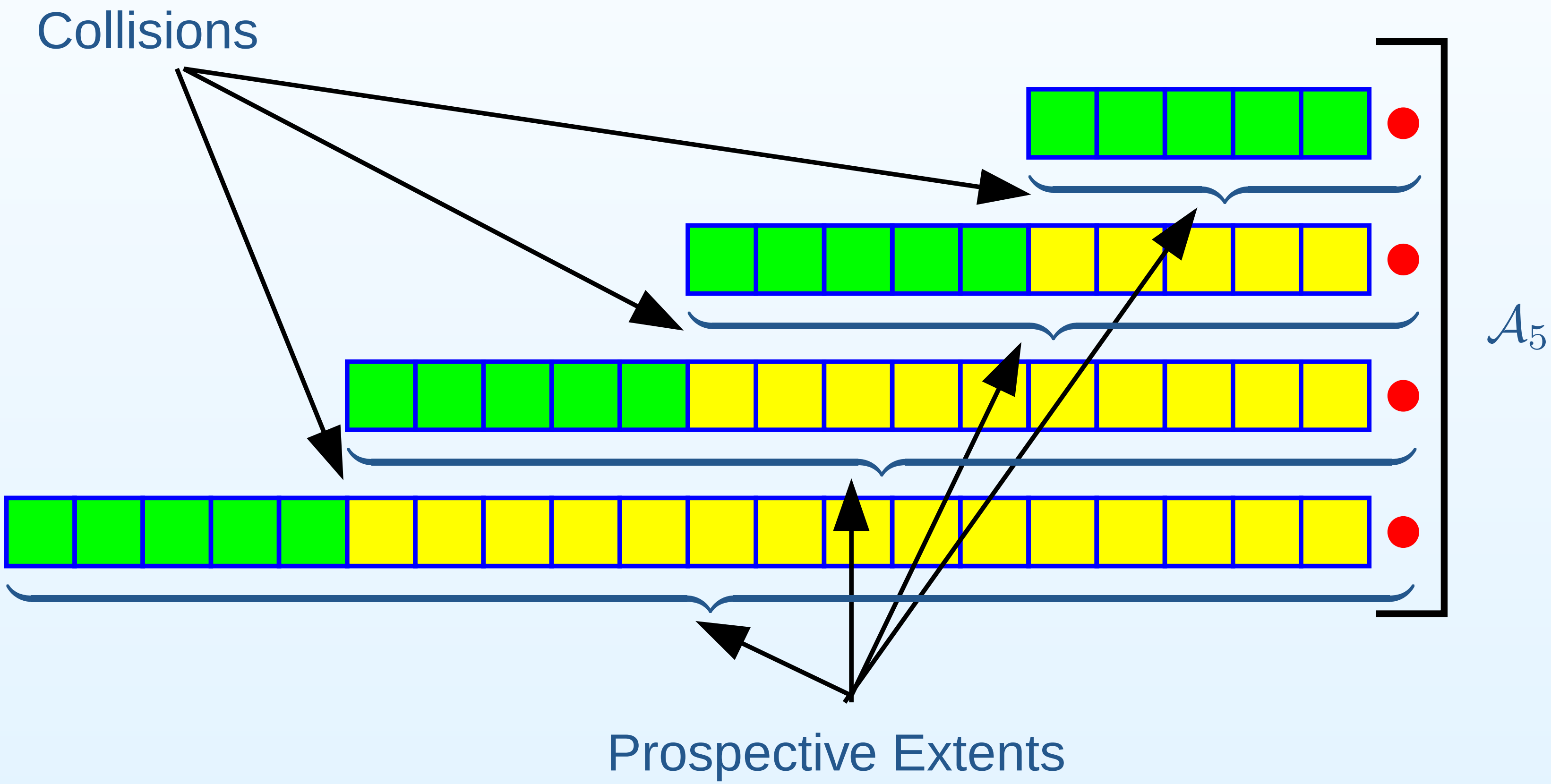

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  - $\mathcal{A}_{t+1}$ contains all the suffixes in $\mathcal{A}_t$ having α_{t+1} *as their* $(t+1)$*-th element*.

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- Otherwise,
  - We select the $(k - l_t)$*-th smallest subsequence* π_{t+1}. from the *multiset*

    $$\mathcal{F}_t = \Big\{ (\sigma_t)^{r_i} c_i \Big| \begin{array}{ll} & T_i \in \mathcal{A}_t \\ \textit{and} & (\sigma_t)^{r_i} c_i \textit{ is the prosp. ext. of } T_i \end{array} \Big\}$$

    using [Blum et al. 1973] (two subsequences in $\mathcal{F}_t$ can be compared in $O(1)$).

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- Otherwise,
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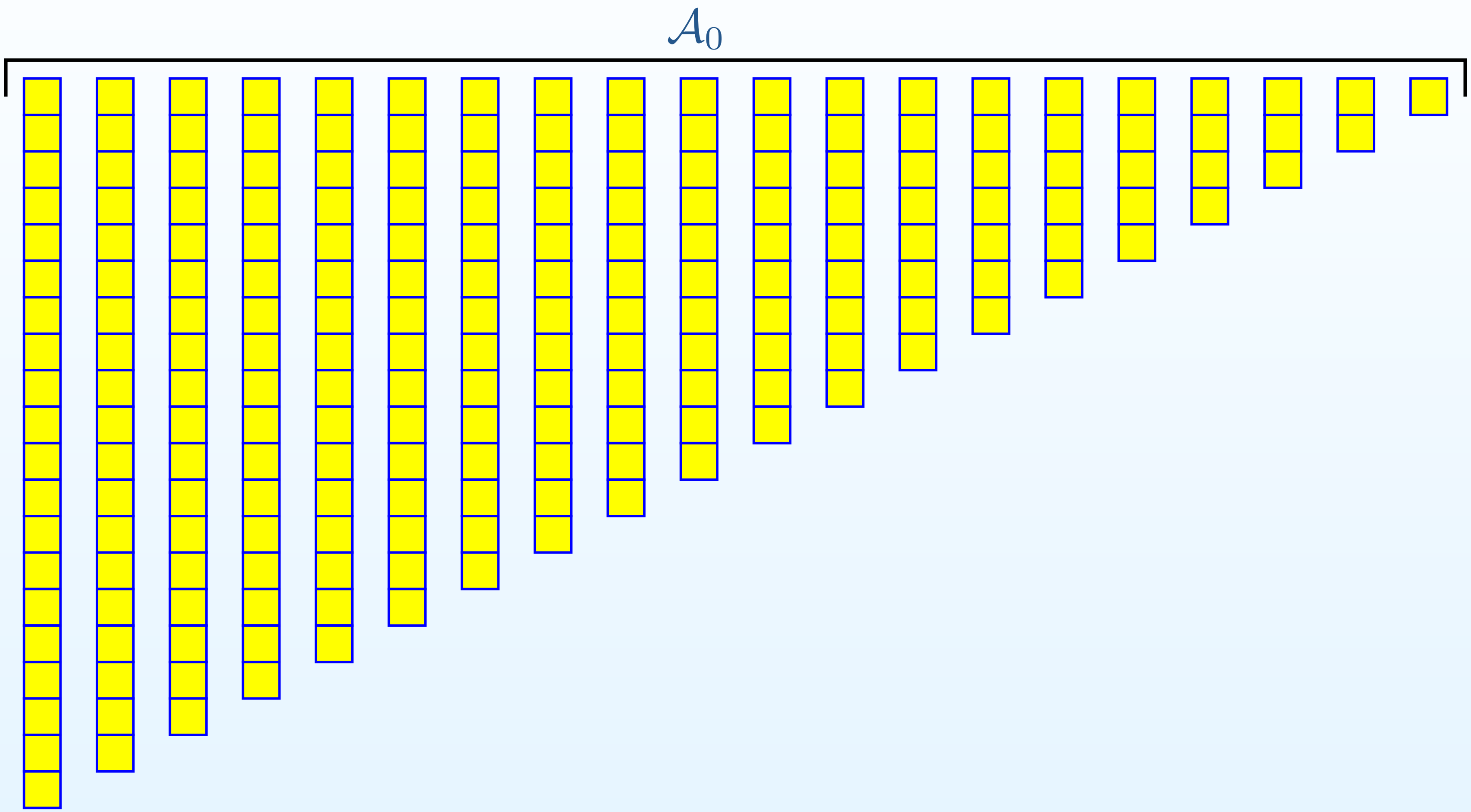
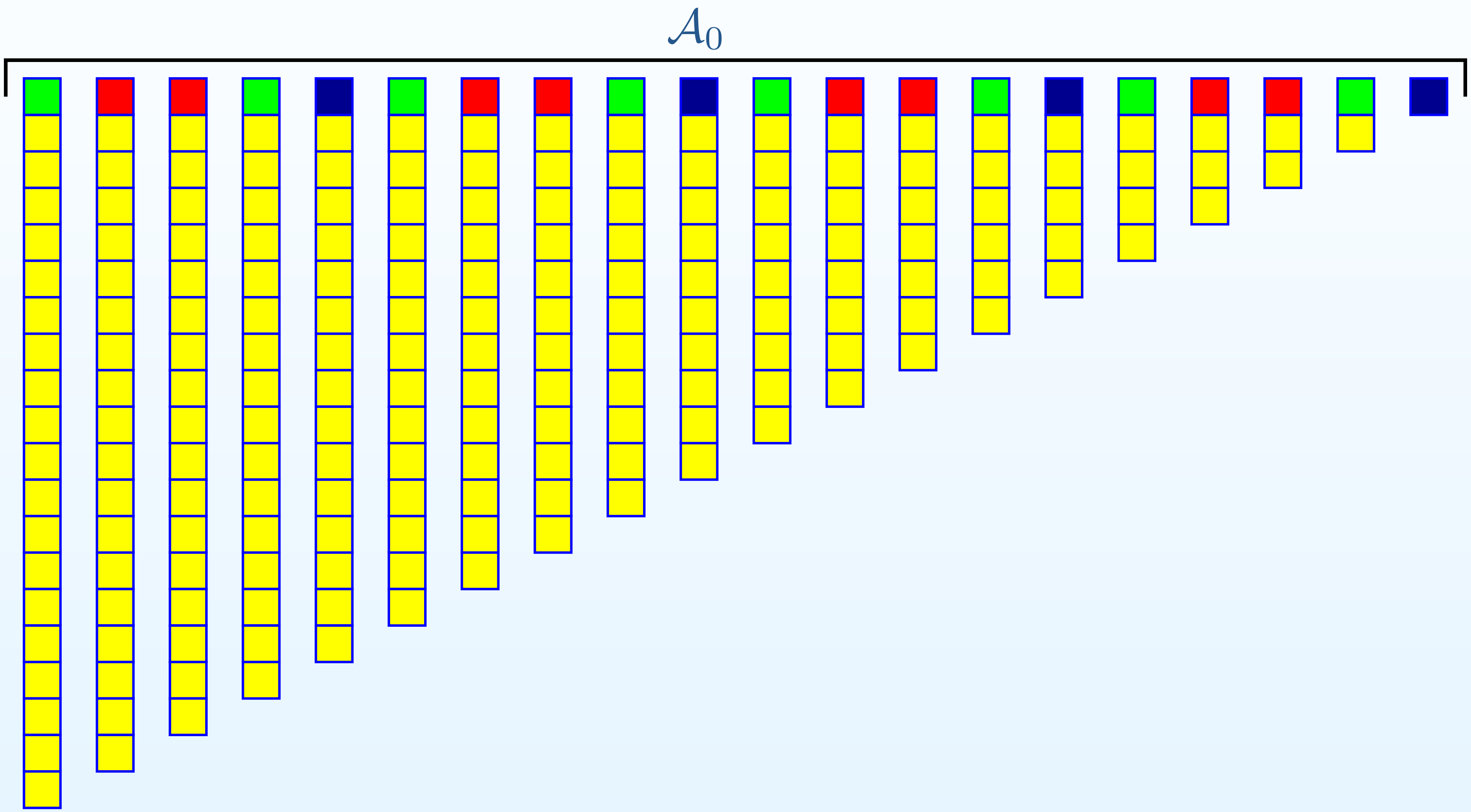
  - We set $\sigma_{t+1} = \pi_{t+1}$.
  - $\mathcal{A}_{t+1}$ contains all the suffixes in $\mathcal{A}_t$ having π_{t+1} *as their extent*.

Suffix selection, second attempt: exploiting collisions



$k = 10$, $l_0 = 0$, Phase 0

Suffix selection, second attempt: exploiting collisions



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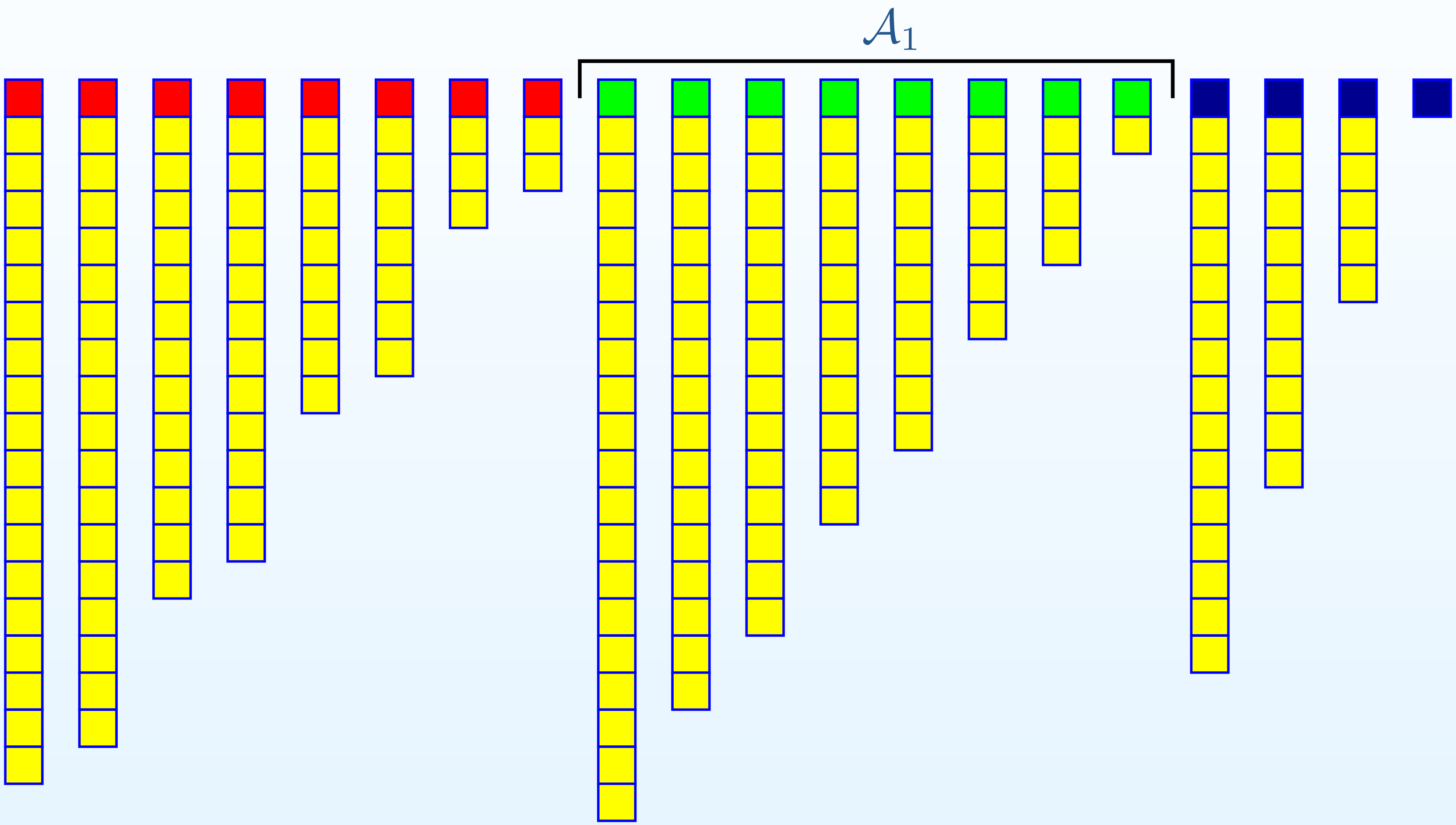
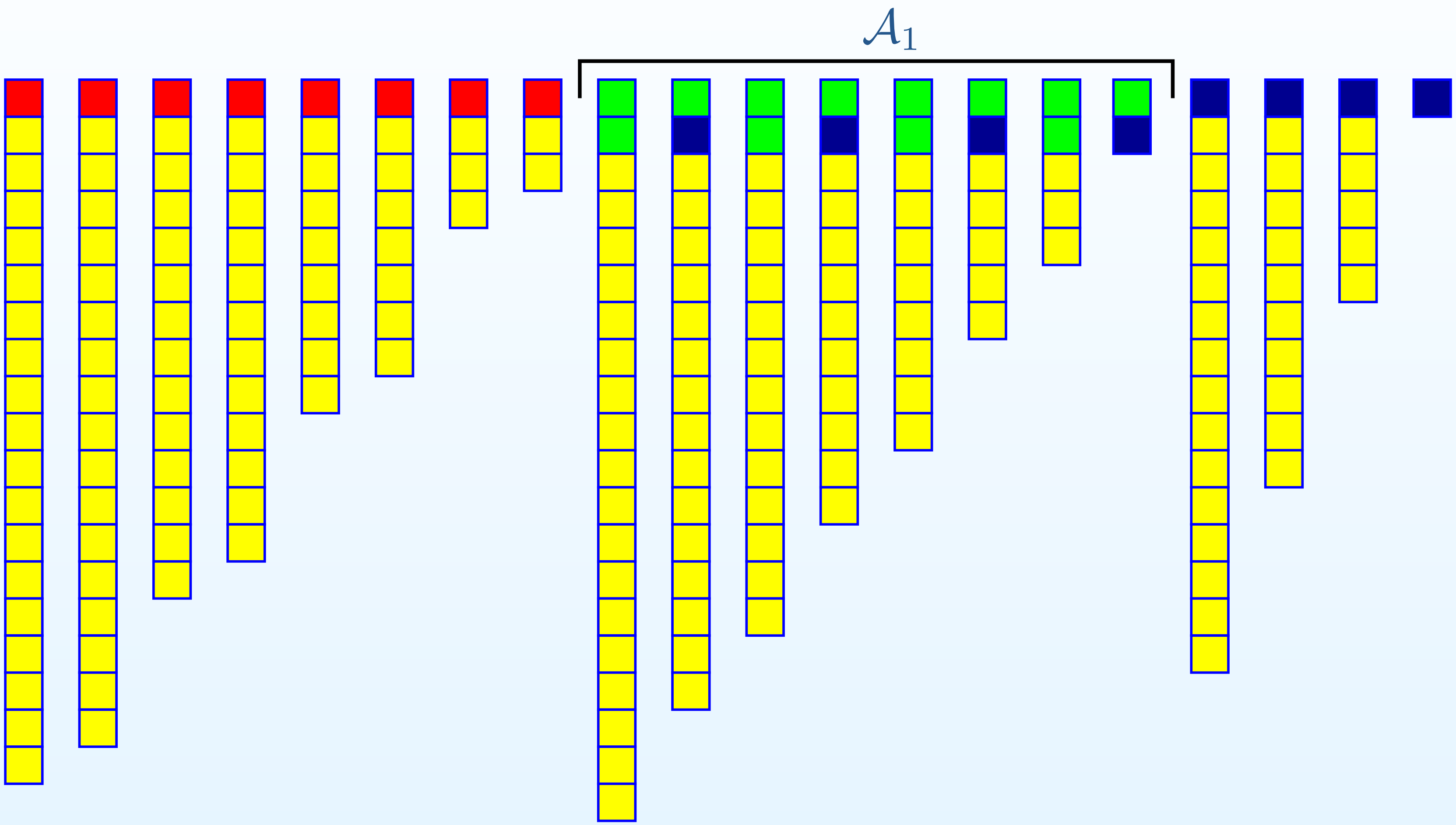
Suffix selection, second attempt: exploiting collisions



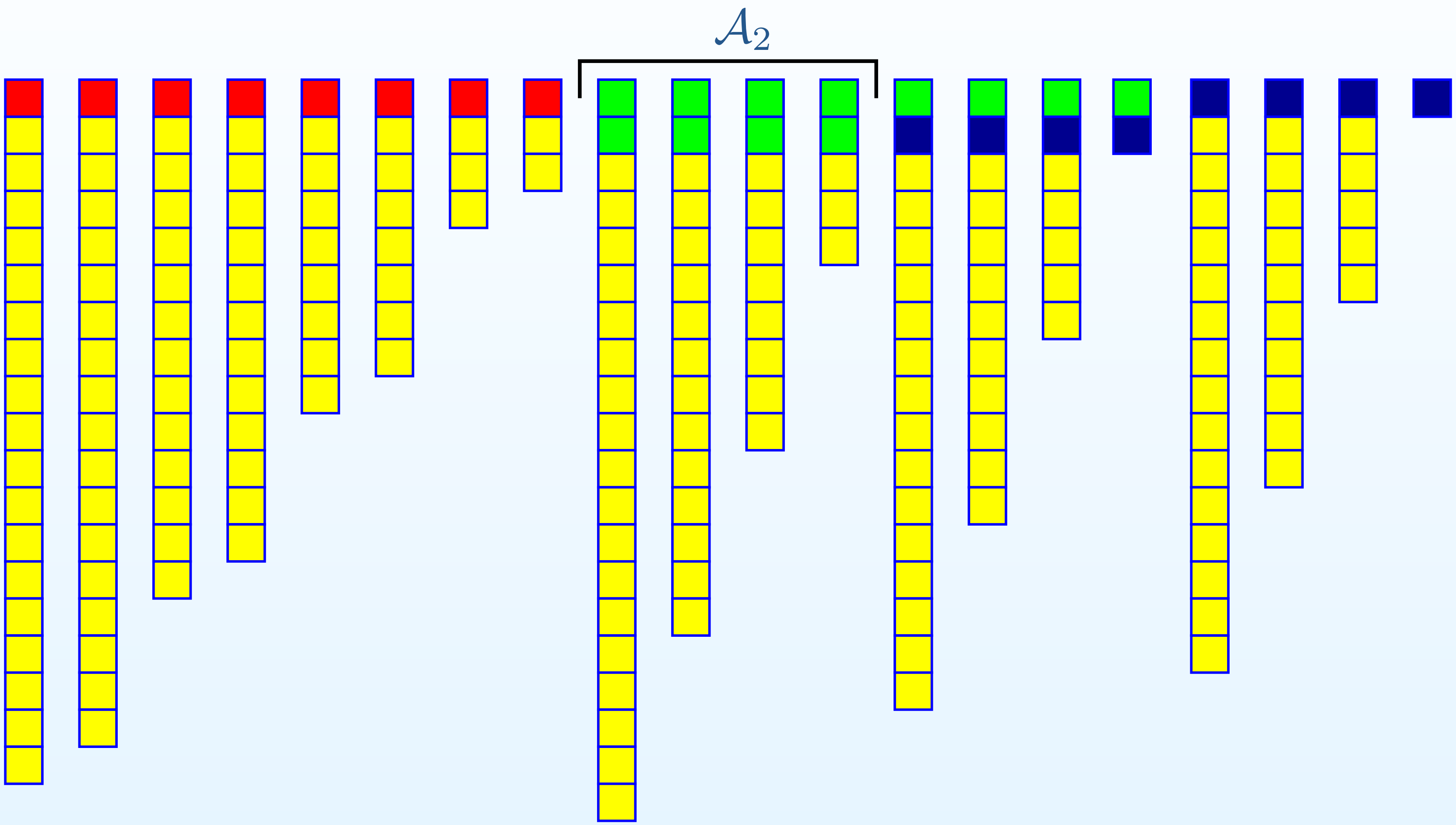
$k = 10$, $l_1 = 8$, Phase 1

Suffix selection, second attempt: exploiting collisions



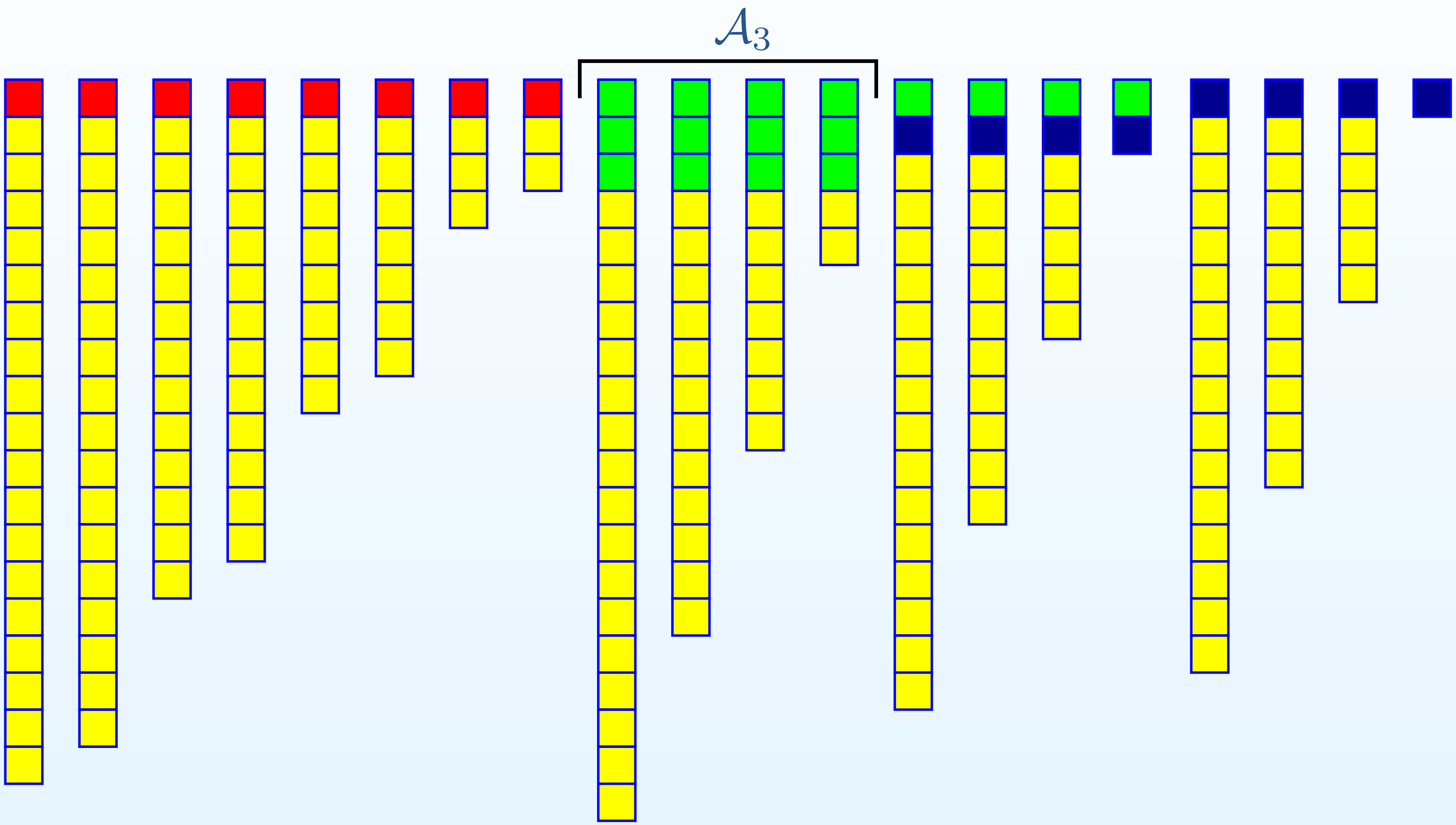
$k = 10$, $l_1 = 8$, Phase 1

Suffix selection, second attempt: exploiting collisions



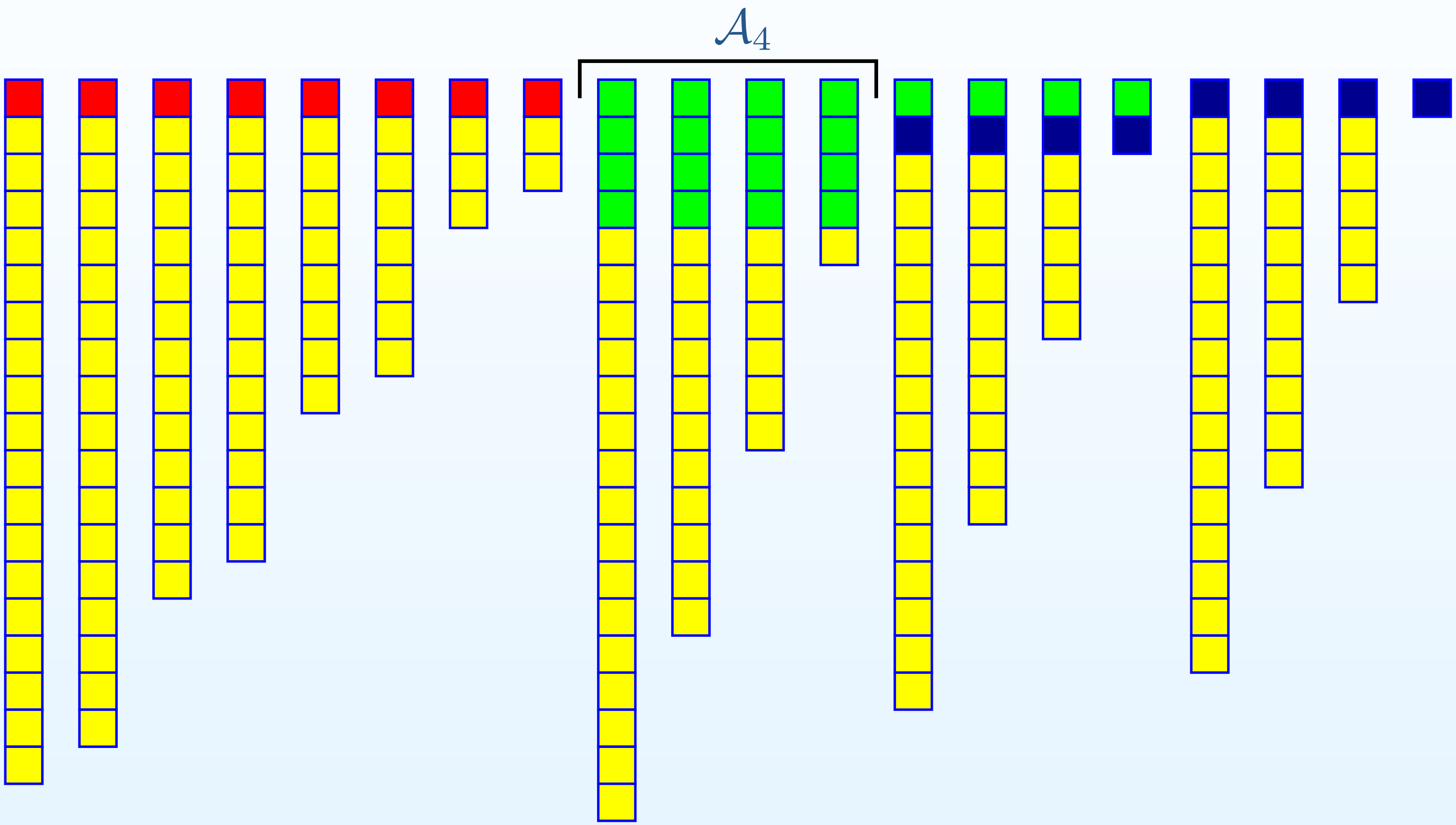
$k = 10$, $l_2 = 8$, Phase 2

Suffix selection, second attempt: exploiting collisions



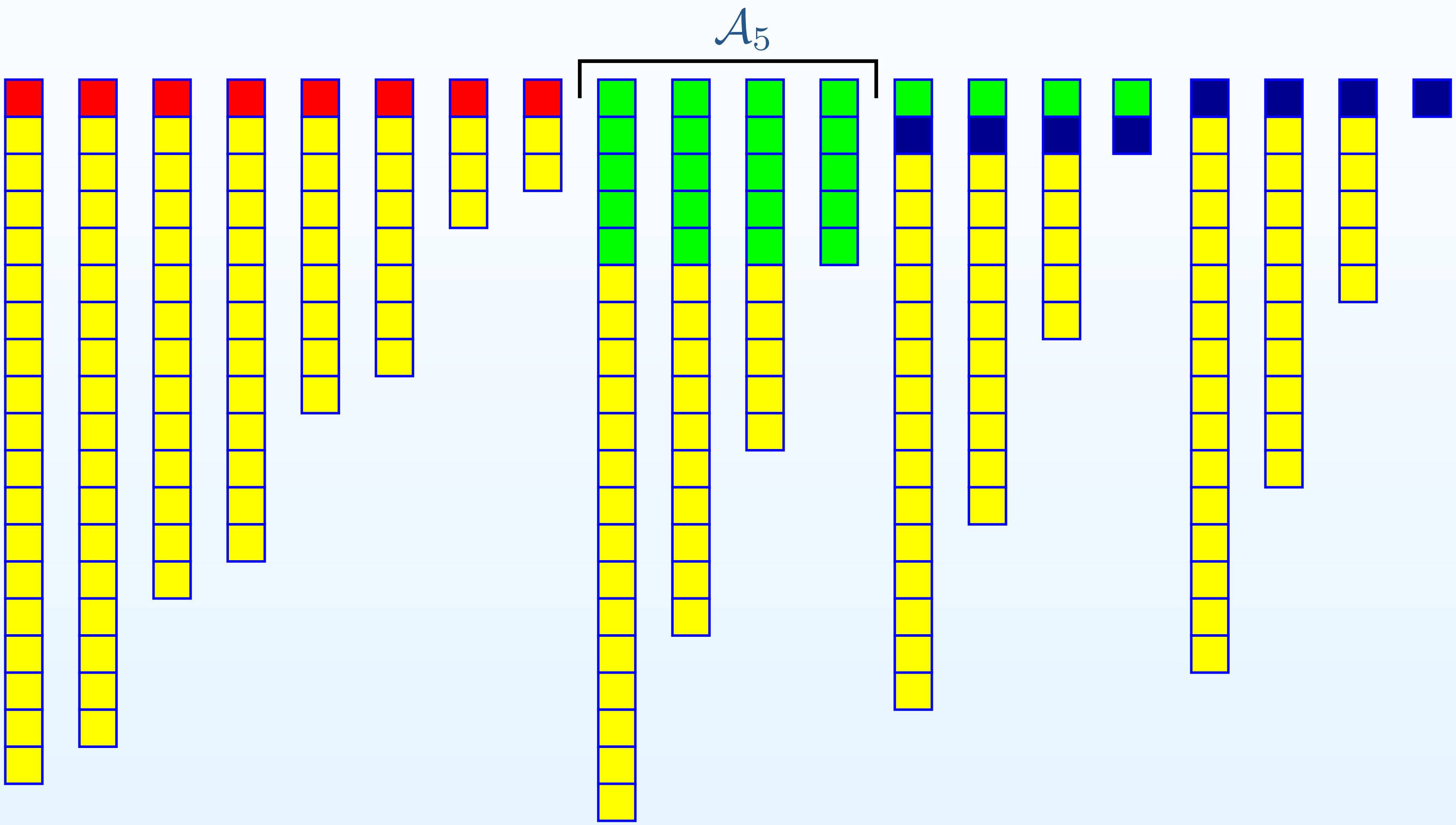
$k = 10$, $l_3 = 8$, Phase 3

Suffix selection, second attempt: exploiting collisions



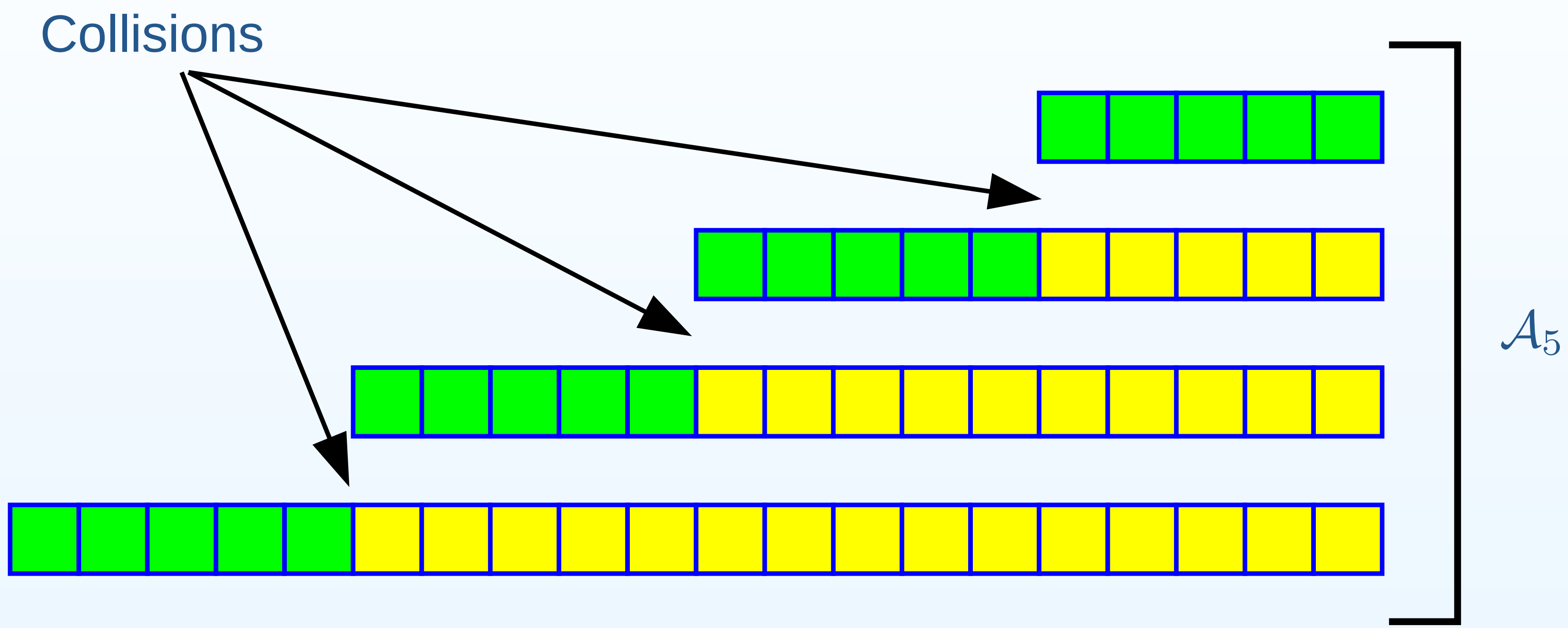
$k = 10$, $l_4 = 8$, Phase 4

Suffix selection, second attempt: exploiting collisions

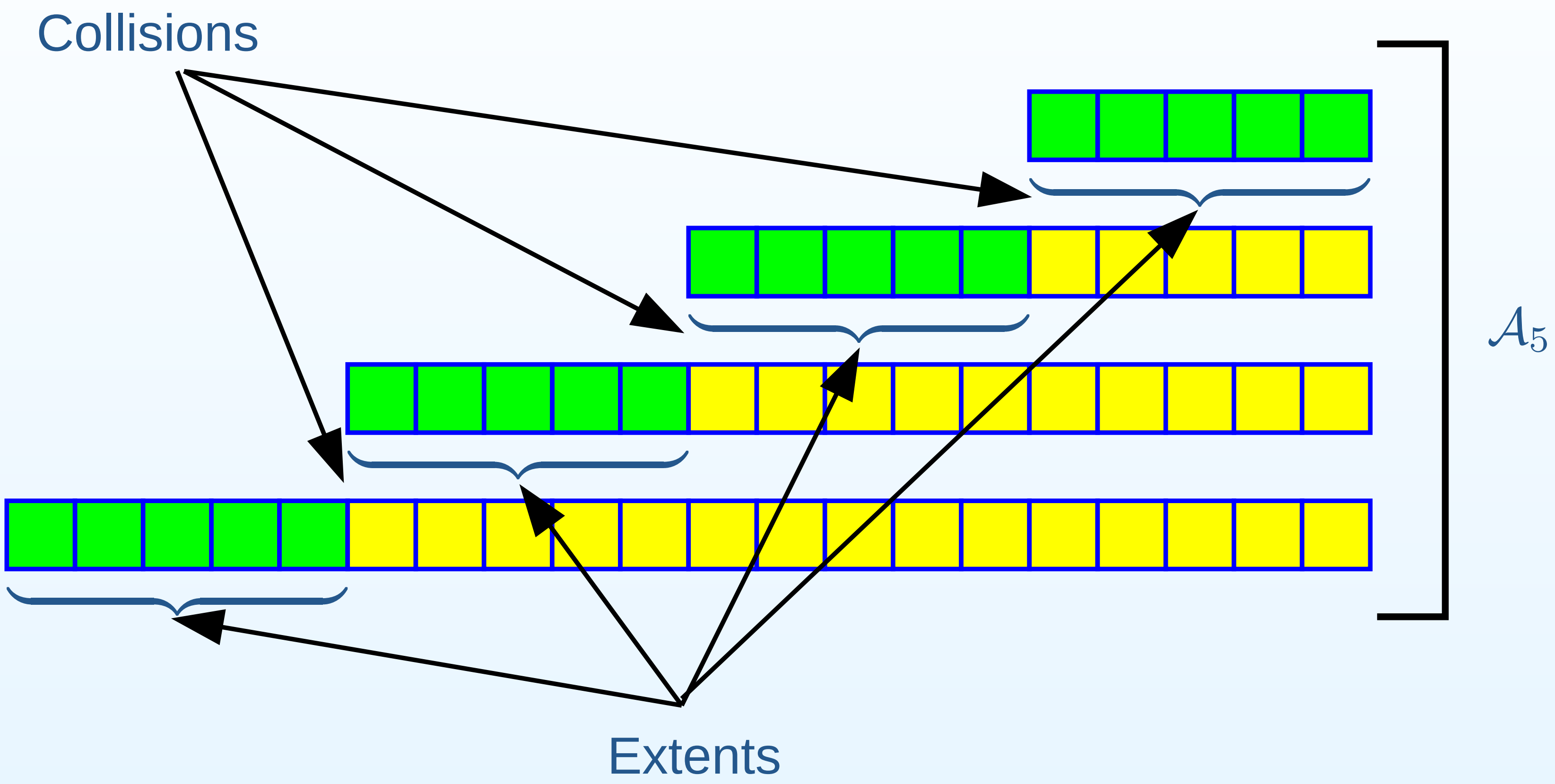


$k = 10$, $l_5 = 8$, Phase 5

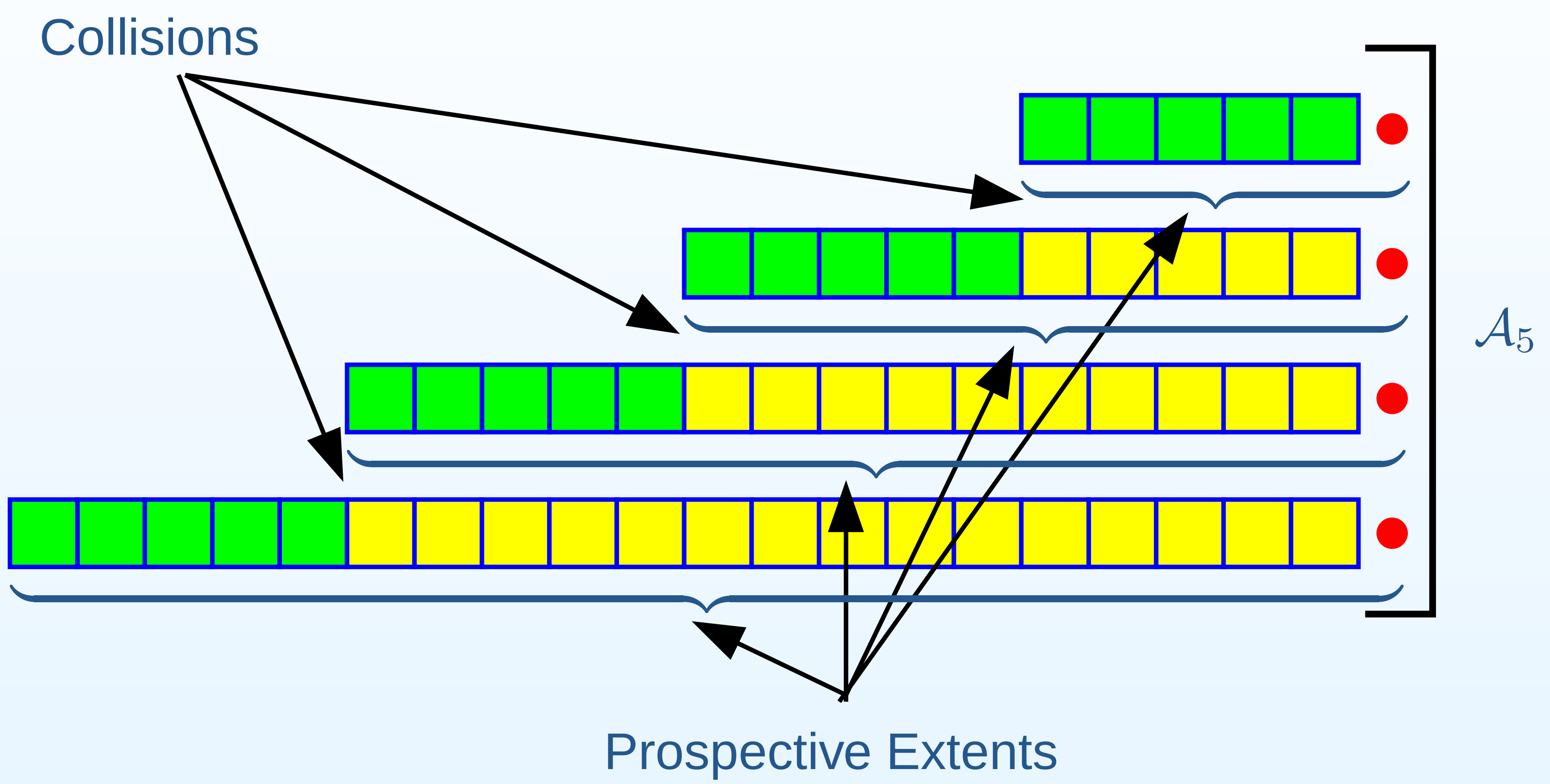
Suffix selection, second attempt: exploiting collisions



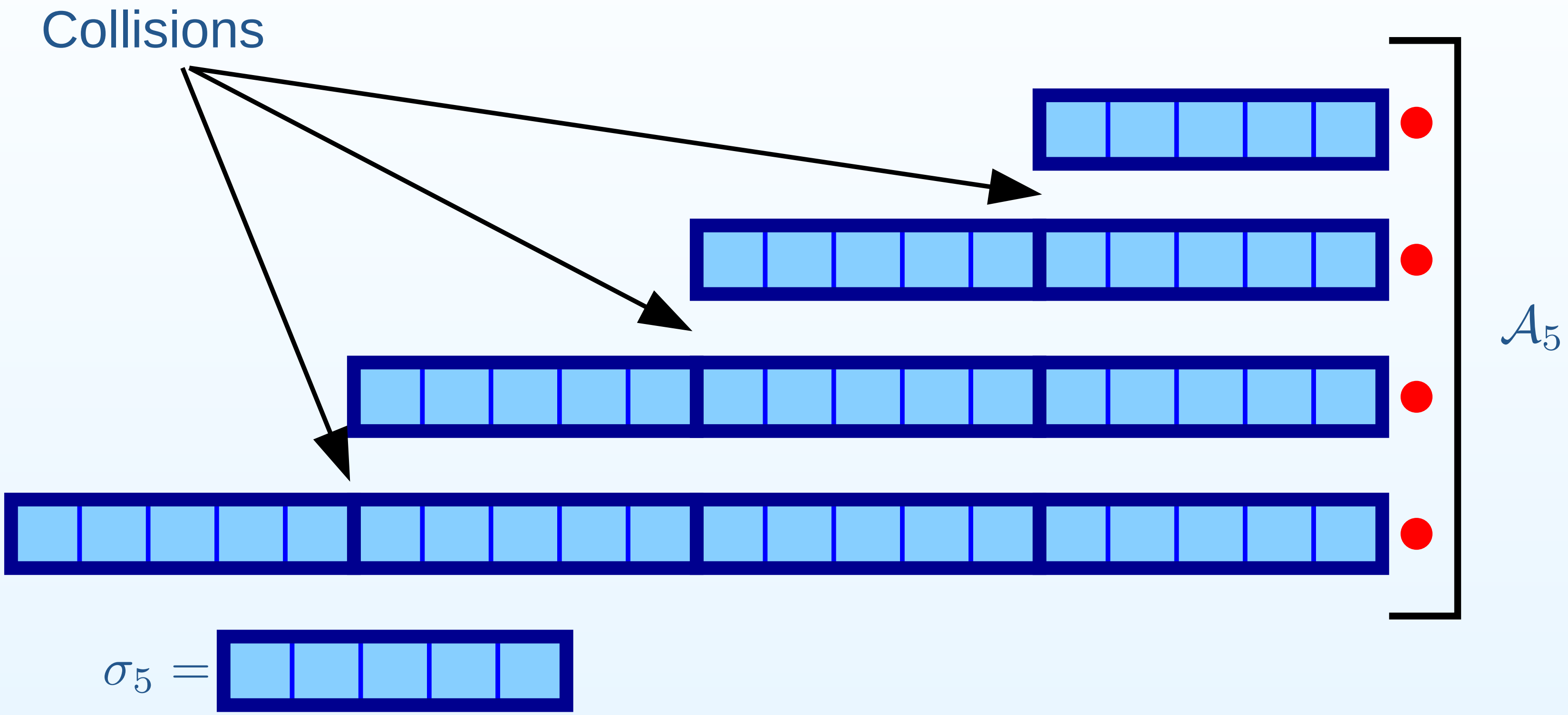
Suffix selection, second attempt: exploiting collisions



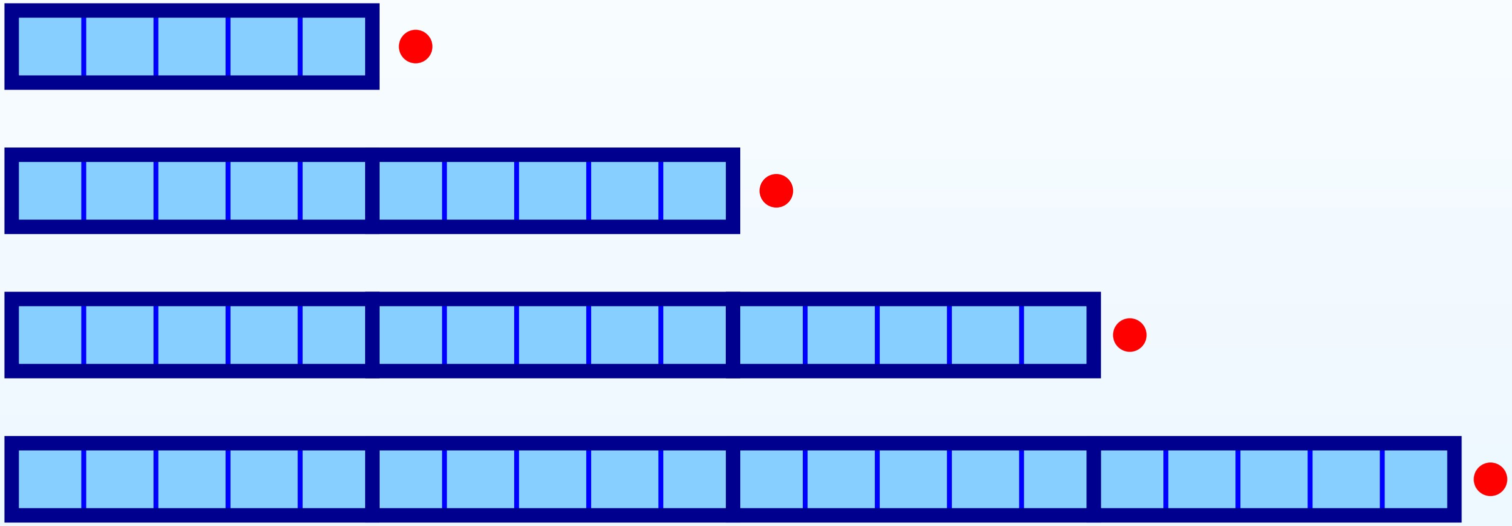
Suffix selection, second attempt: exploiting collisions



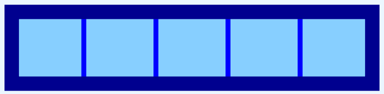
Suffix selection, second attempt: exploiting collisions



Suffix selection, second attempt: exploiting collisions

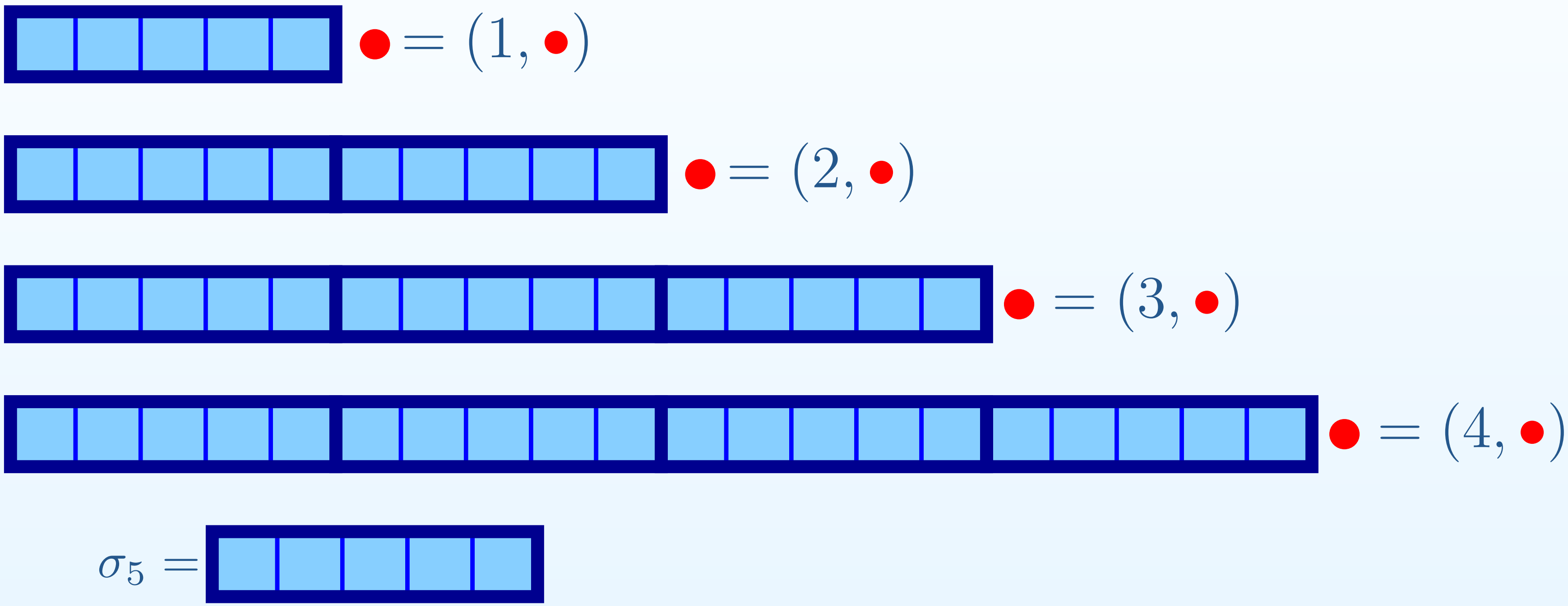


$\sigma_5 =$



How do we compare the prospective extents in multiset $\mathcal{F}_t$?

Suffix selection, second attempt: exploiting collisions



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- Each subsequence $(\sigma_t)^{r_i} c_i$ can be represented *by the pair* (r_i, c_i) (integer/element pair).

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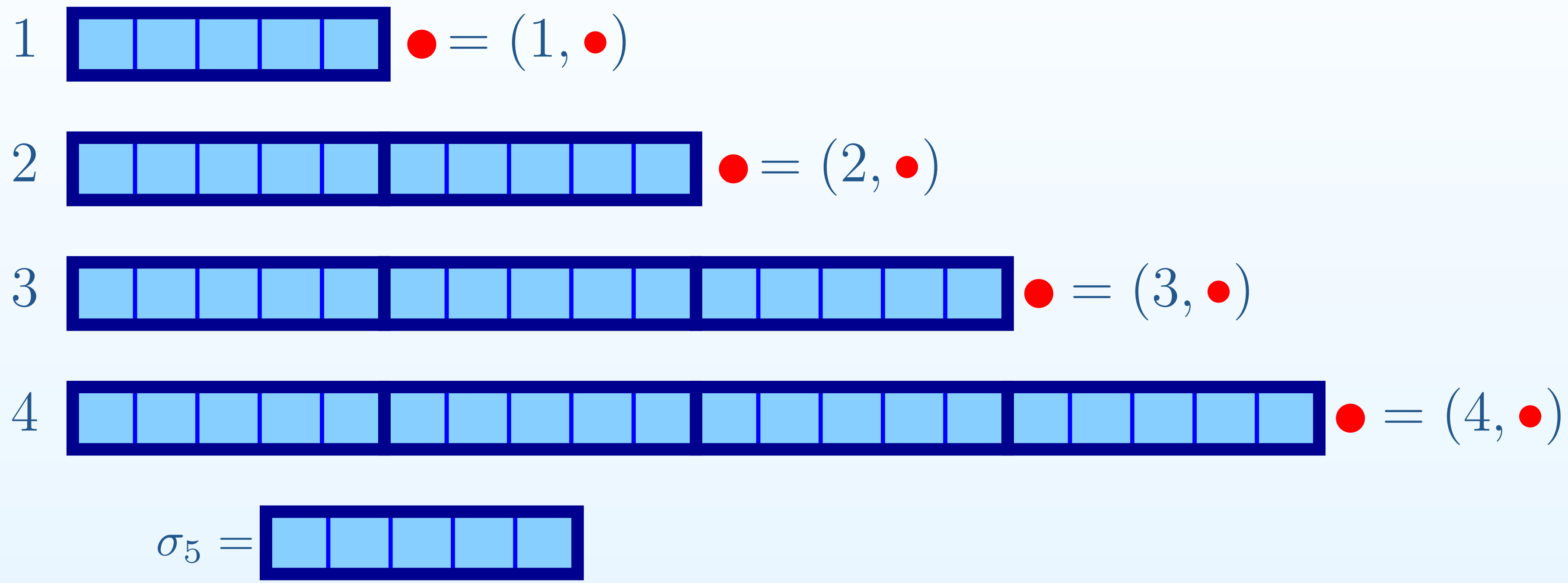

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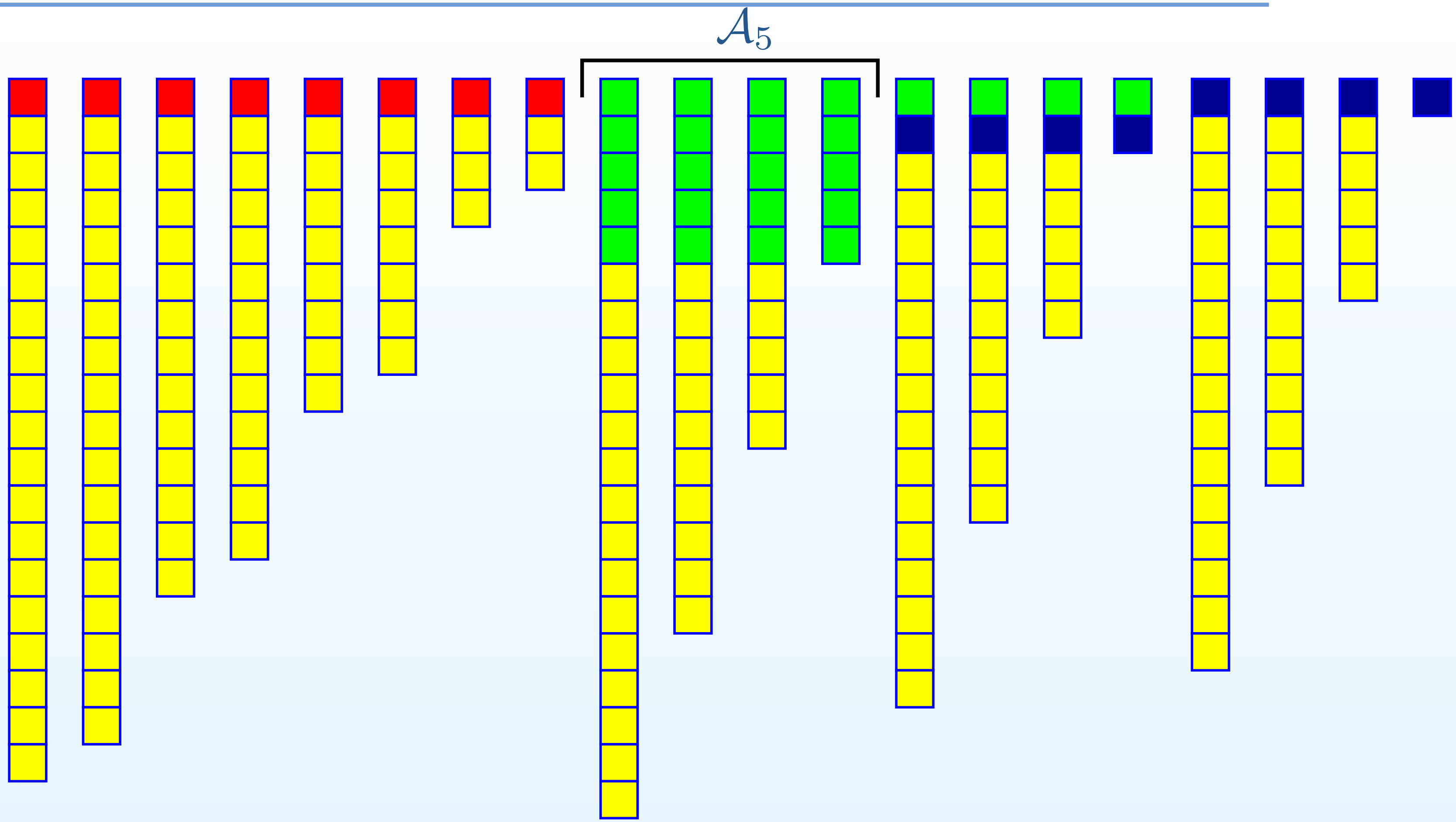
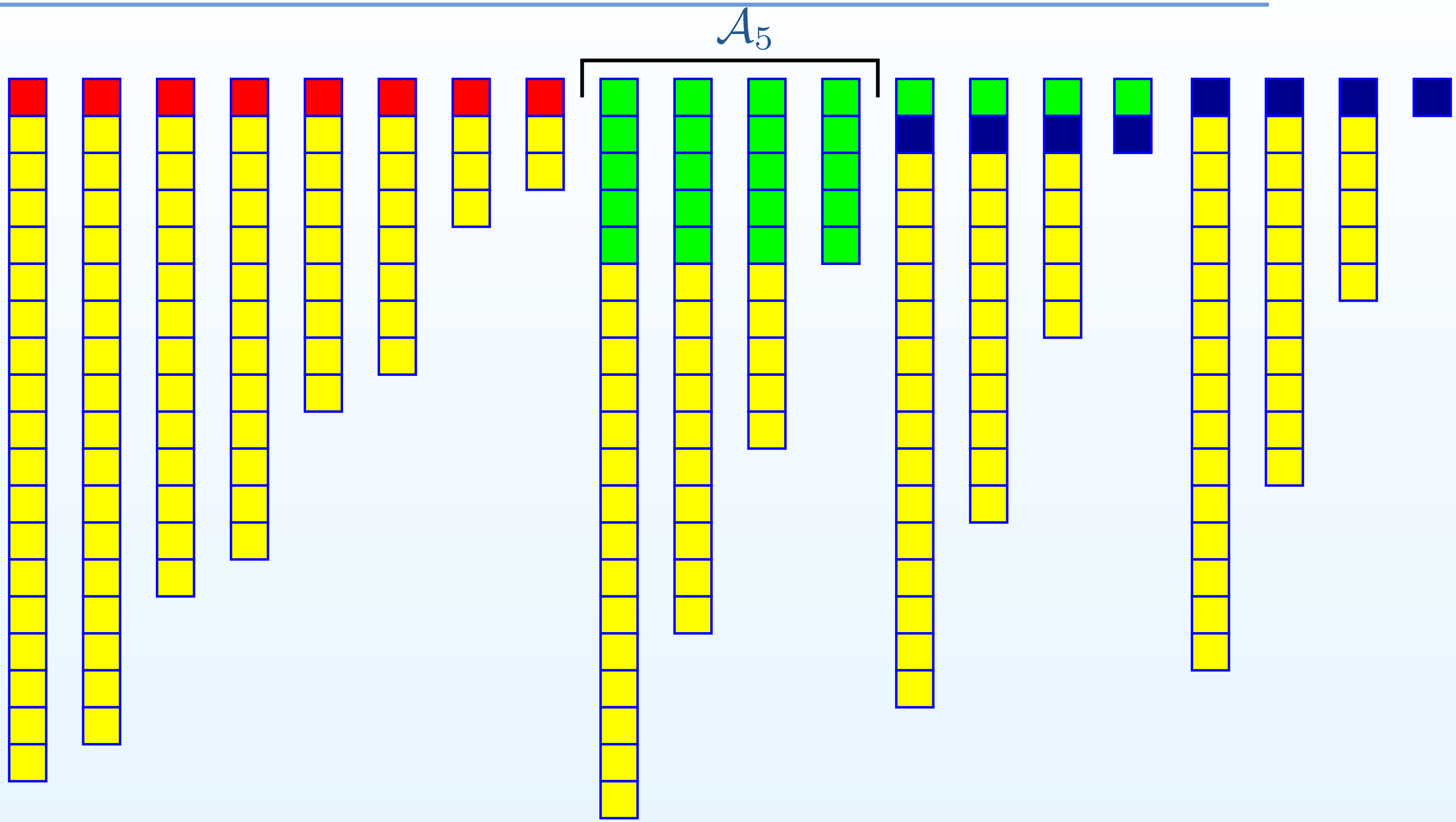
- Each subsequence $(\sigma_t)^{r_i} c_i$ can be represented *by the pair* (r_i, c_i) (integer/element pair).
- To compare two subsequences in $\mathcal{F}_t$ *we can just use their pairs*.

Suffix selection, second attempt: exploiting collisions



$k = 10$, $l_5 = 8$, Phase 5

Suffix selection, second attempt: exploiting collisions



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Now we want the active suffix *with the prosp ext. of rank* $(k - l_5) = 2\ldots$

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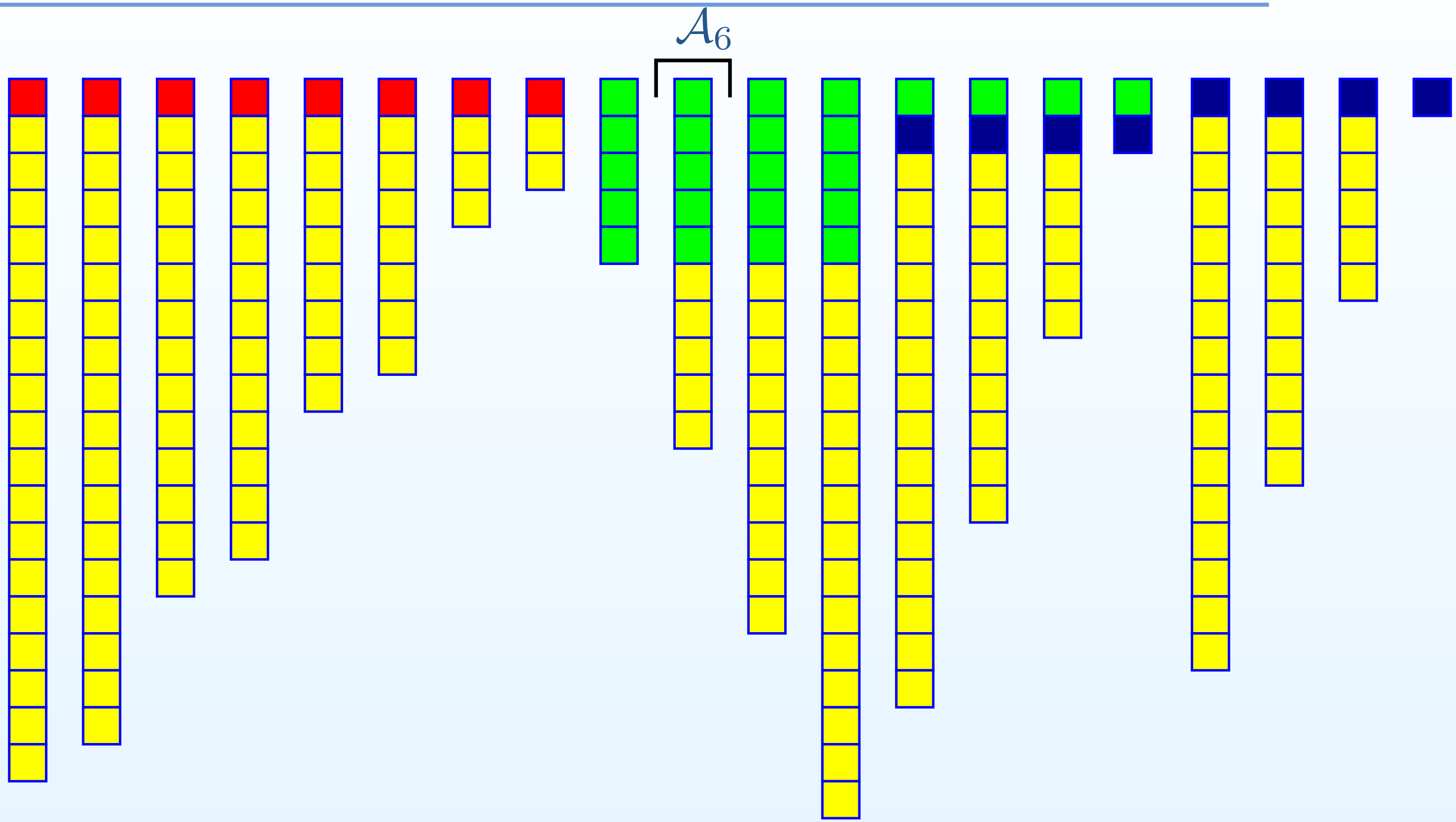

$k = 10$, $l_6 = 9$, Phase 6

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. . . exploiting the collisions in $\mathcal{A}_5$, *we find it in just one Phase Transition*.

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- In the example we went *from the* 11 *phases of the first attempt to just* 6.
- *The complexity of the second algorithm is* $O(n \log n)$ *in the worst case.*
  - Let us *group the phases into macro-phases* $m_0, m_1, \ldots, m_w$ such that, *for any* m_i *and any phase* $t \in m_i$, we have that
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  - For any phase t, the extents of the active suffixes *do not overlap*...
  - ...and phase t has at most n/σ_t *active suffixes*.
  - Therefore, *the cost of each macro-phase is* $O(n)$ and the final $O(n \log n)$ bound follows immediately.

Suffix selection, second attempt: exploiting collisions

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We need one last step:

> *We have to be able to reuse the work done on inactive suffixes.*

Suffix selection, third attempt: reusing inactive suffixes

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- The central issue is *how much the extent of an active suffix T_i in phase $t+1$ is enlarged during the transition from phase t*.
- What do we add to the extent of T_i in the second solution?
  - (a) *All the extents of the active suffixes that follow T_i and collide with it.*
  - (b) *The element c next to the extent of the rightmost suffix in the collision.*

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The challenge now is to avoid these multiple accesses.

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In the third solution the *prospective extent* of T_i is composed by the following:

(a) *All the extents of the active suffixes following* T_i *and colliding with it* (although they don't collide as nicely as in the second solution, as we will see).

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In the third solution the *prospective extent* of T_i is composed by the following:

(a) *All the extents of the active suffixes following* T_i *and colliding with it* (although they don't collide as nicely as in the second solution, as we will see).

(b) *The extent of the forward suffix* f *of the rightmost suffix* r *in collision with* T_i (r is T_i itself if T_i is not in a collision).

Suffix selection, third attempt: reusing inactive suffixes

Let's consider an *active suffix* T_i *in phase* t that remains active after the *transition to phase* $t+1$.

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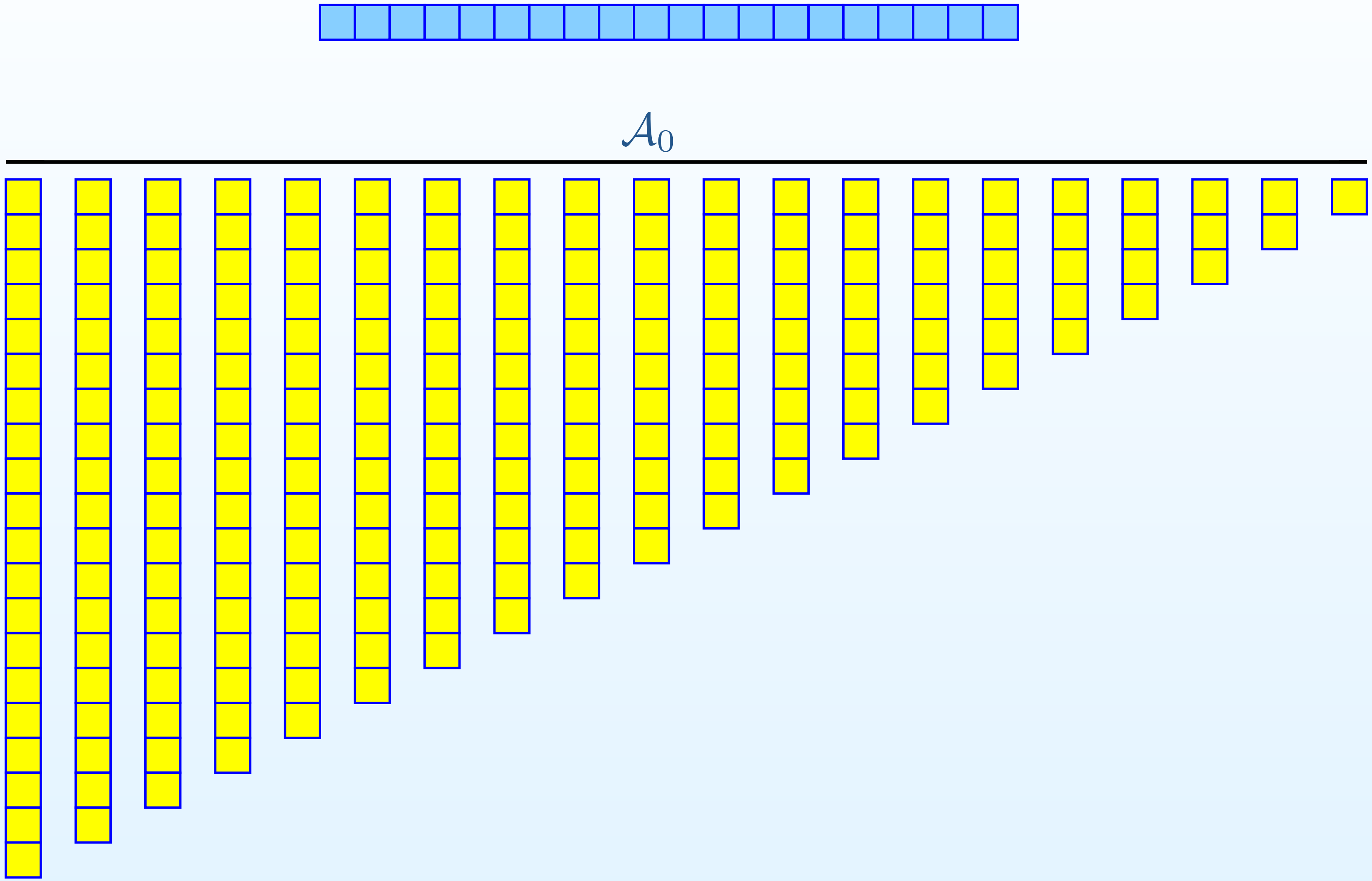

$k = 10$, $l_0 = 0$, Phase 0

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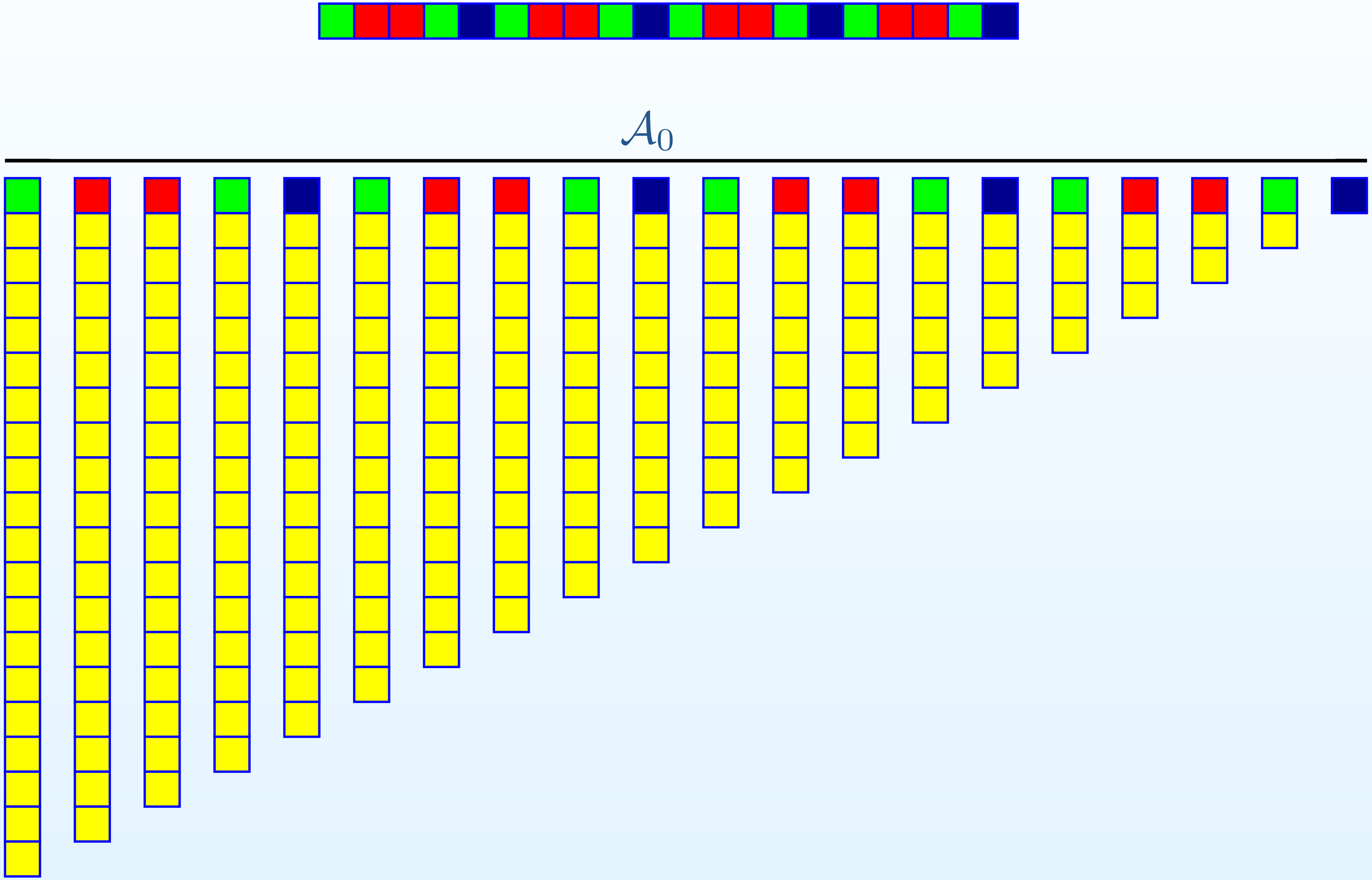

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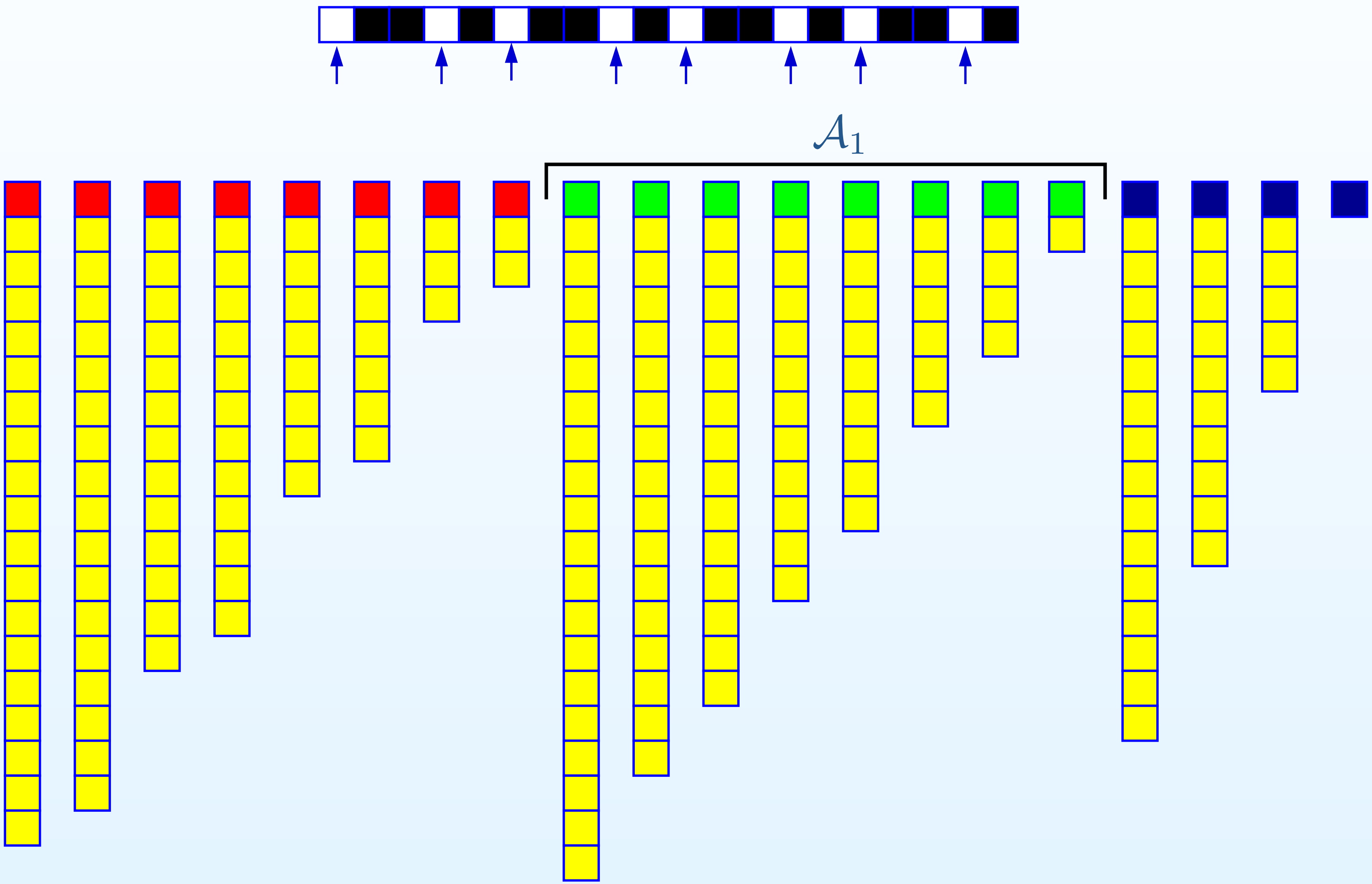

$k = 10$, $l_1 = 8$, Phase 1

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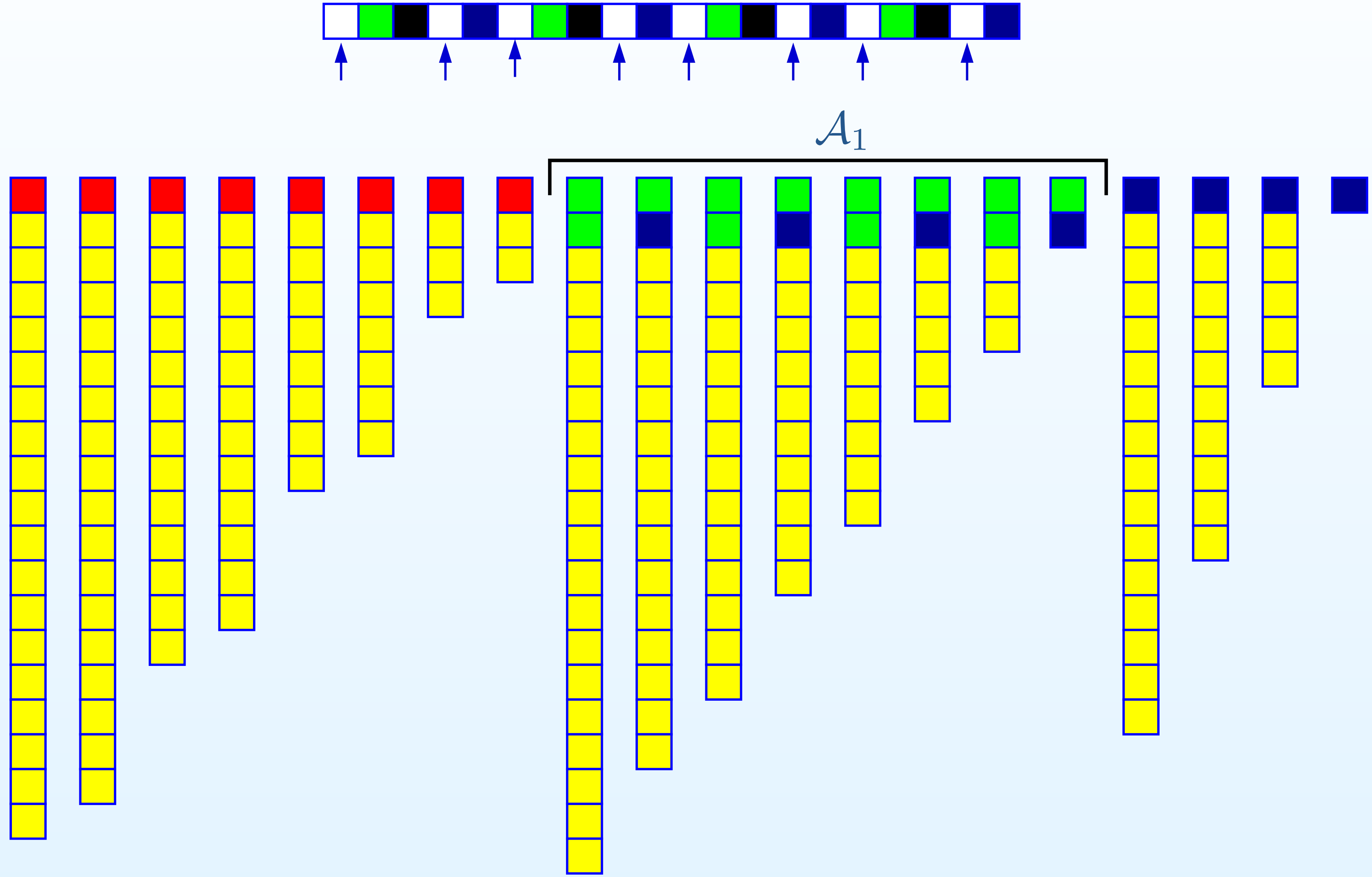

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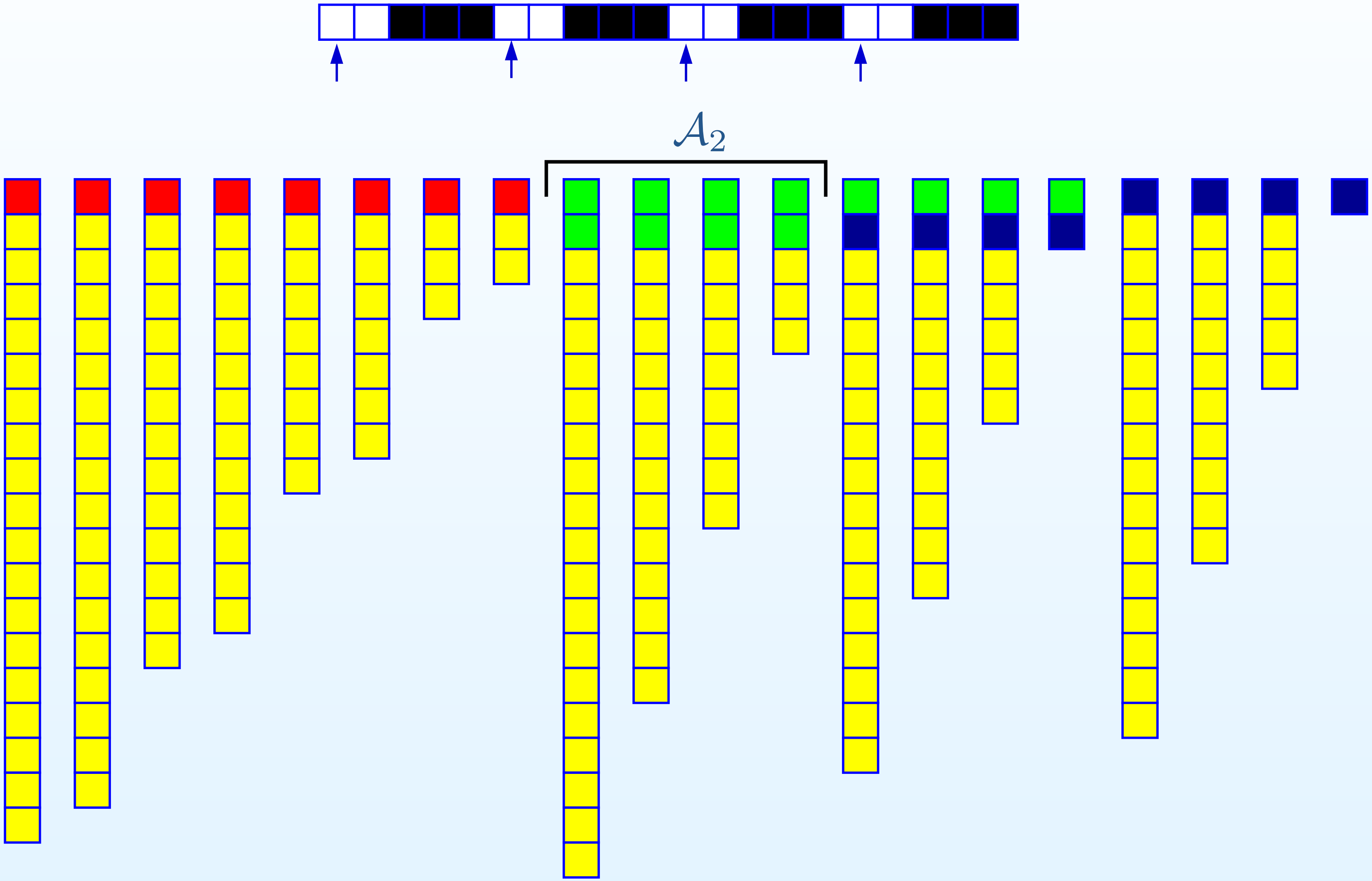
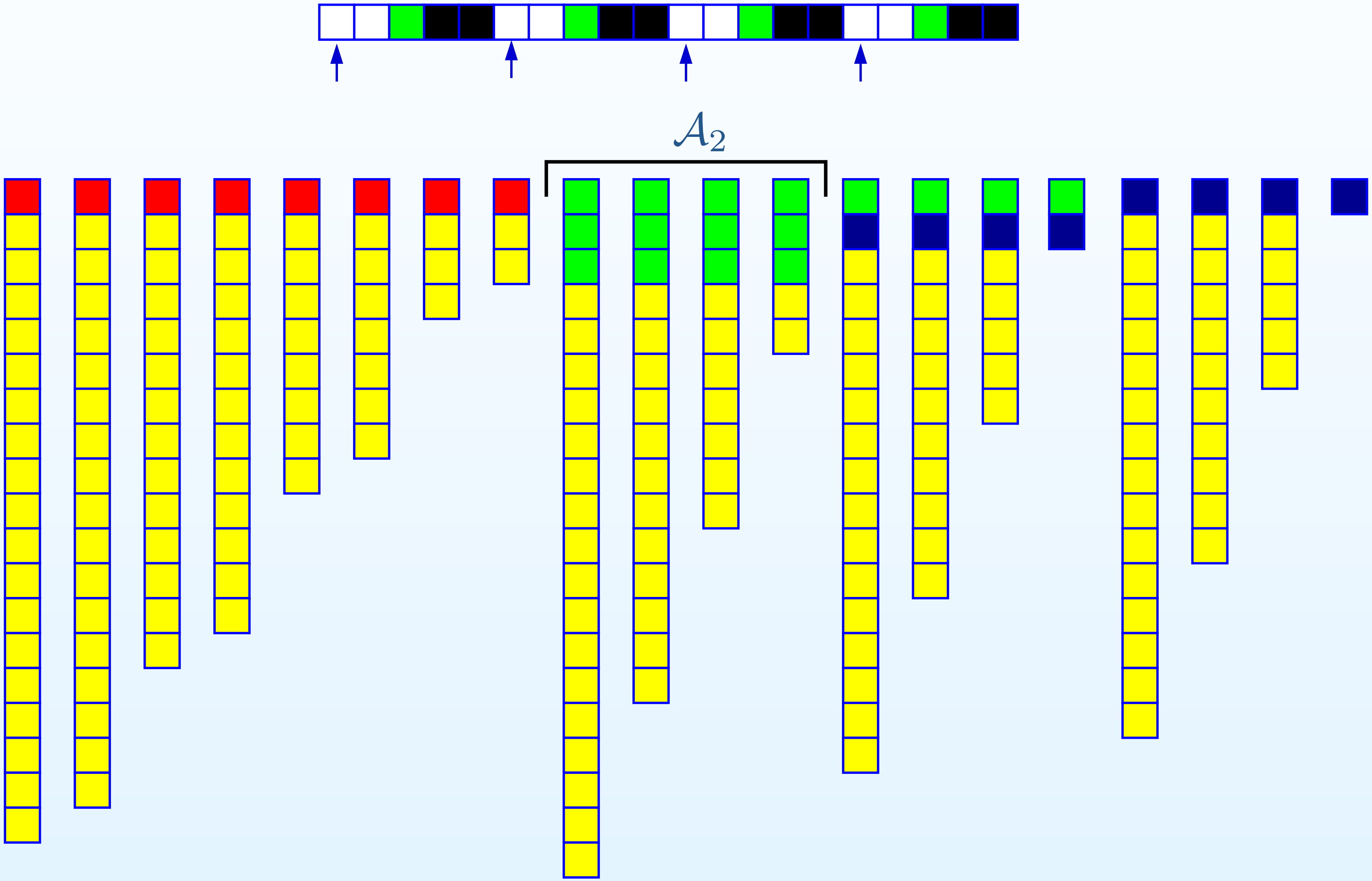


$k = 10$, $l_2 = 8$, Phase 2

Suffix selection, third attempt: reusing inactive suffixes



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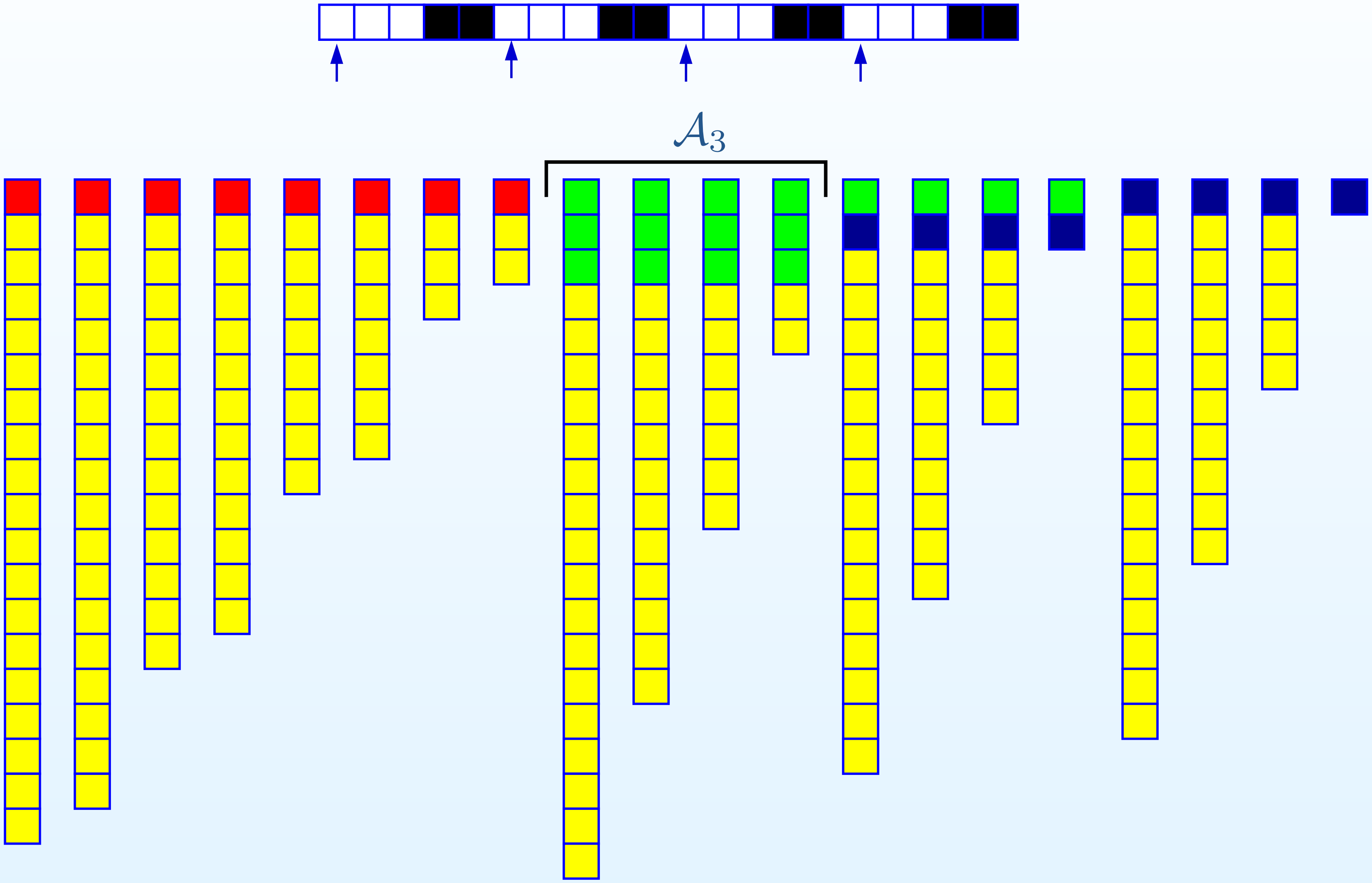

$k = 10$, $l_3 = 8$, Phase 3

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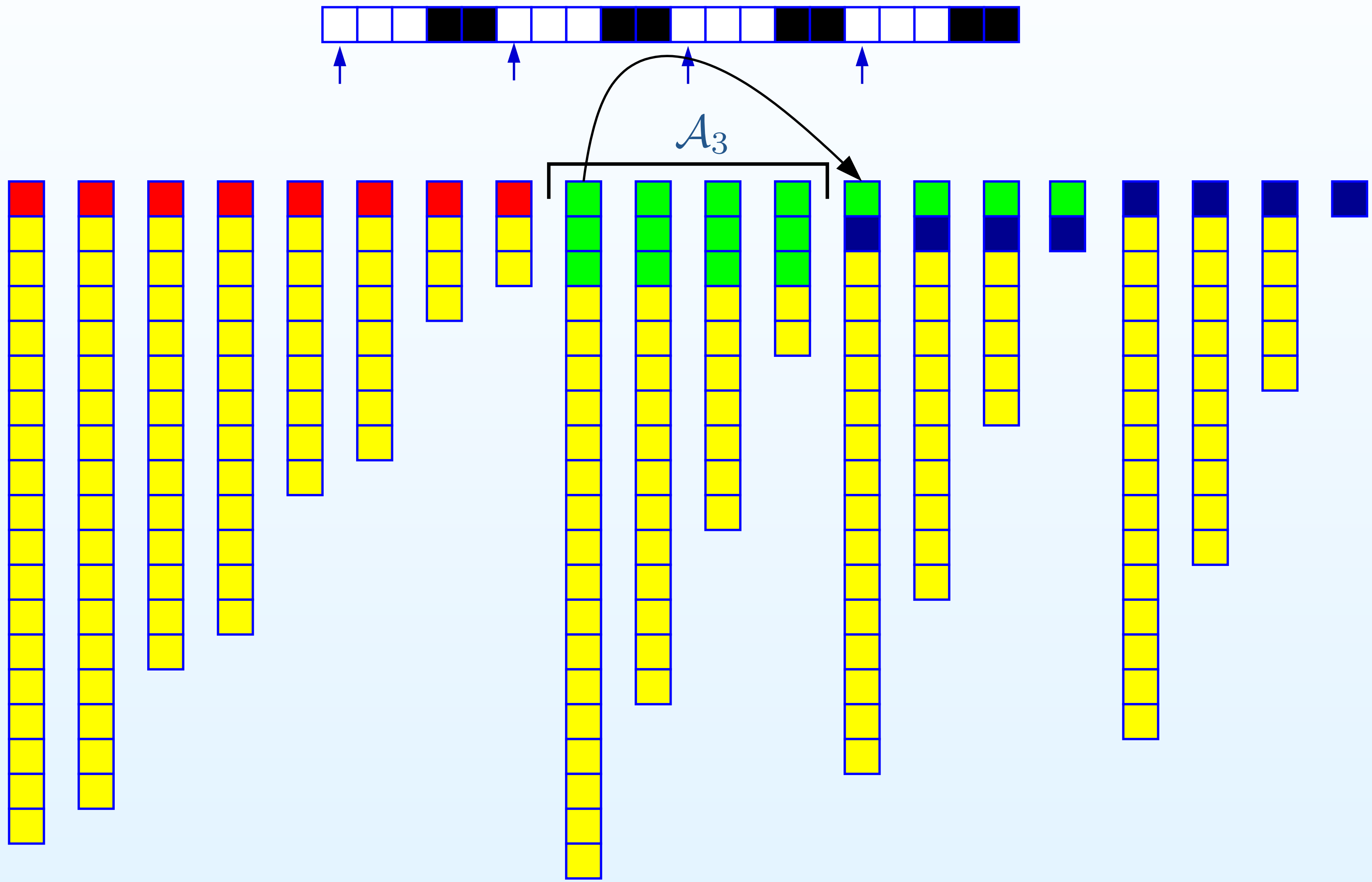

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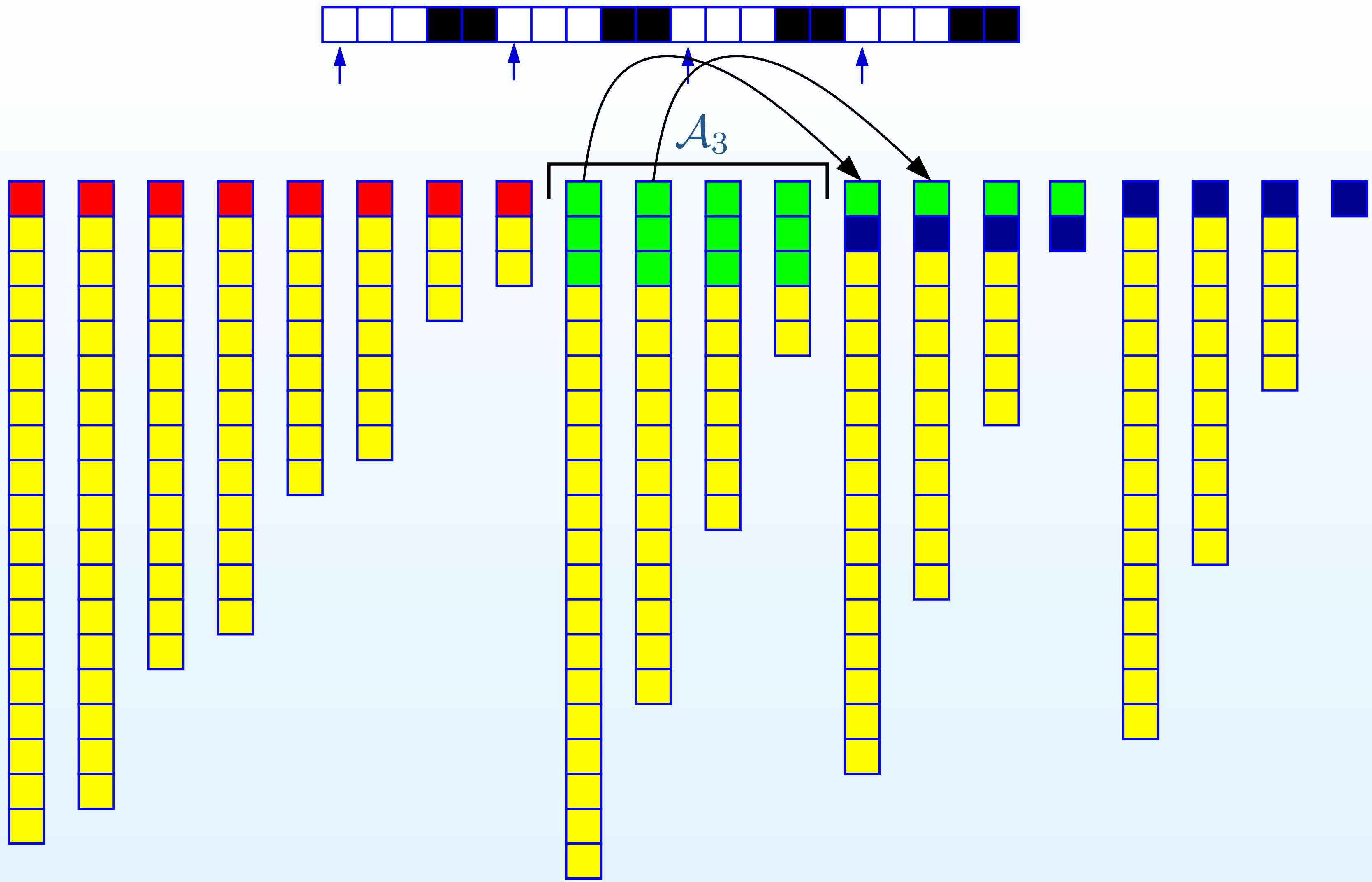

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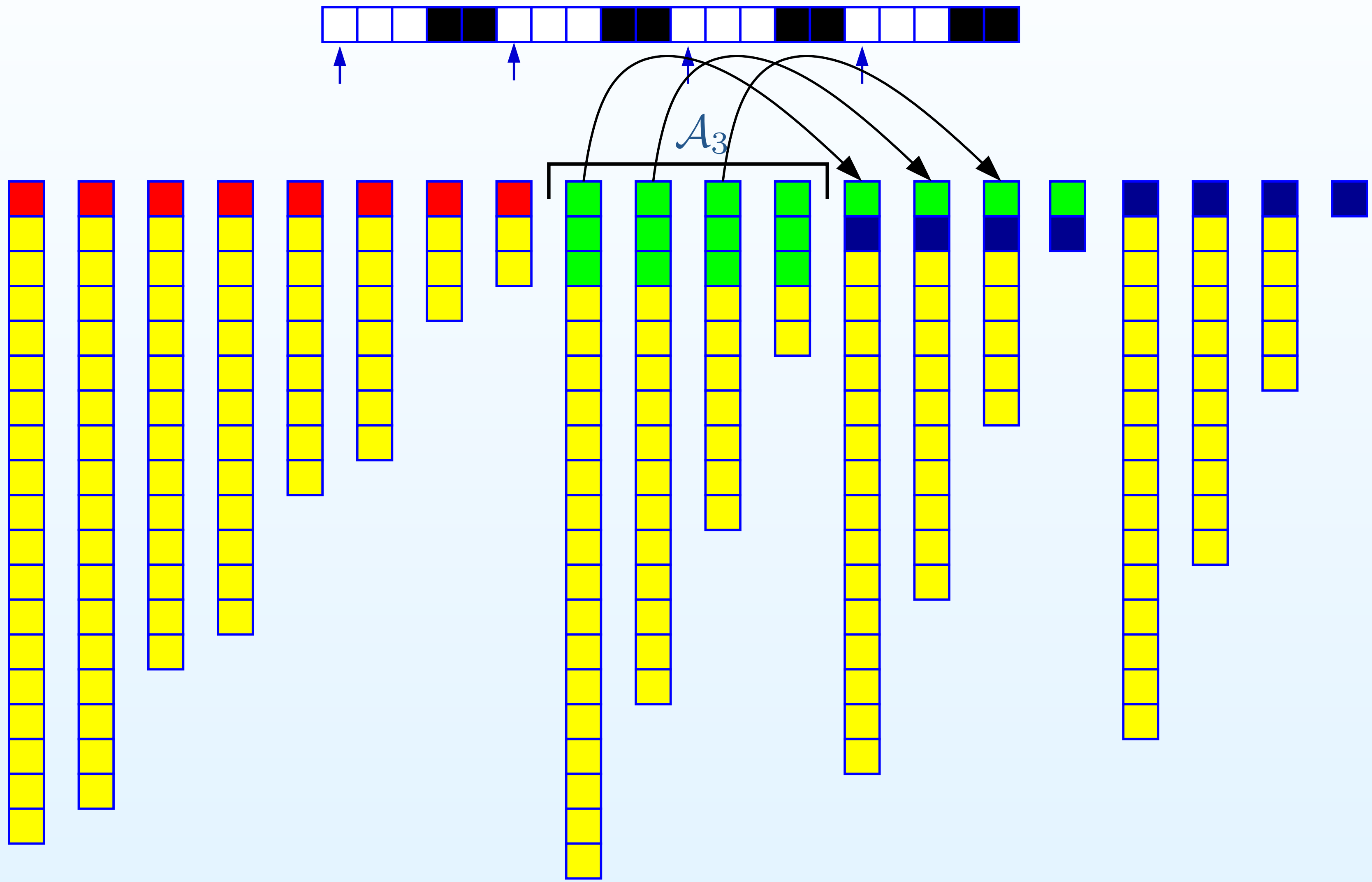

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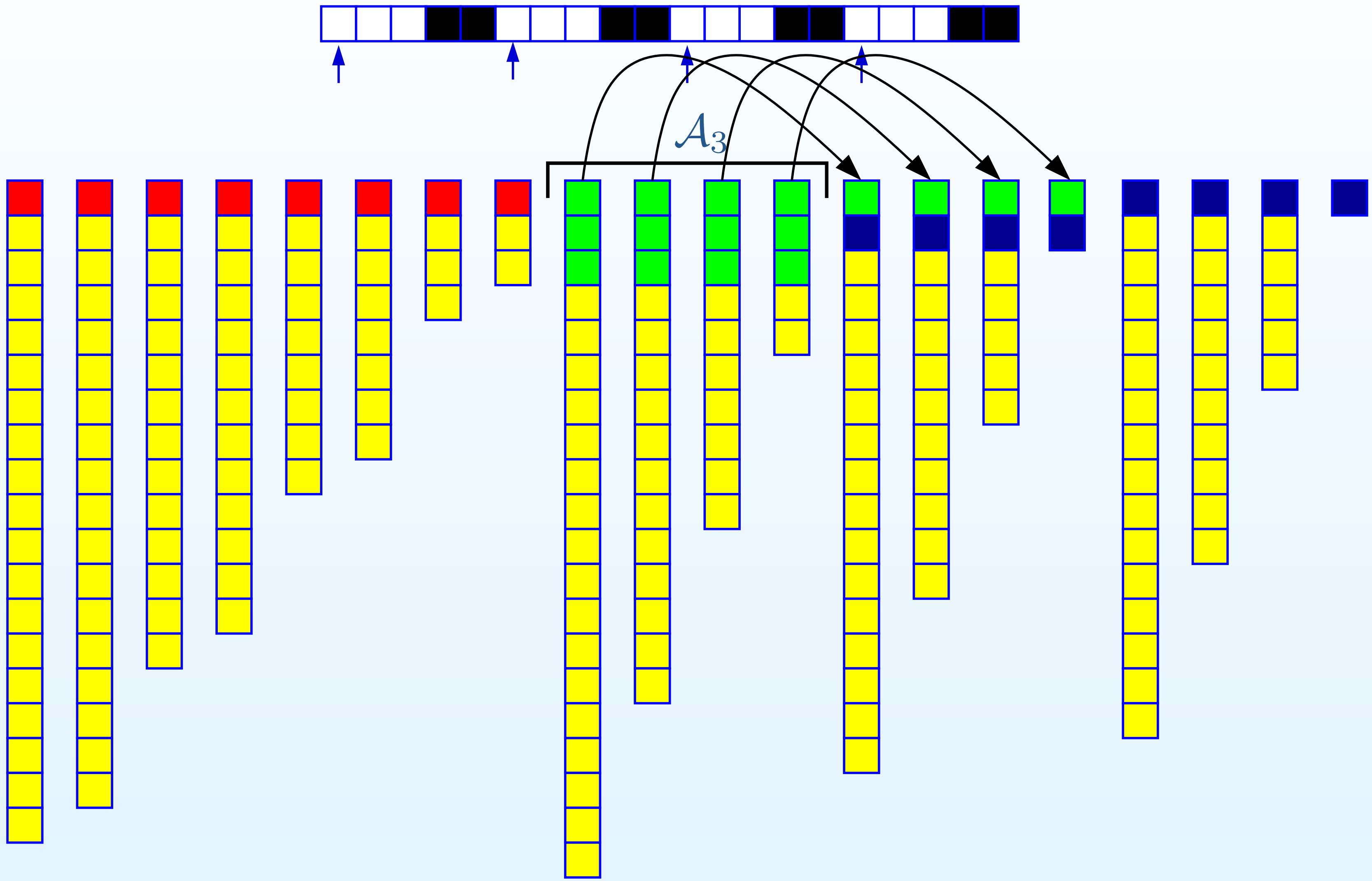

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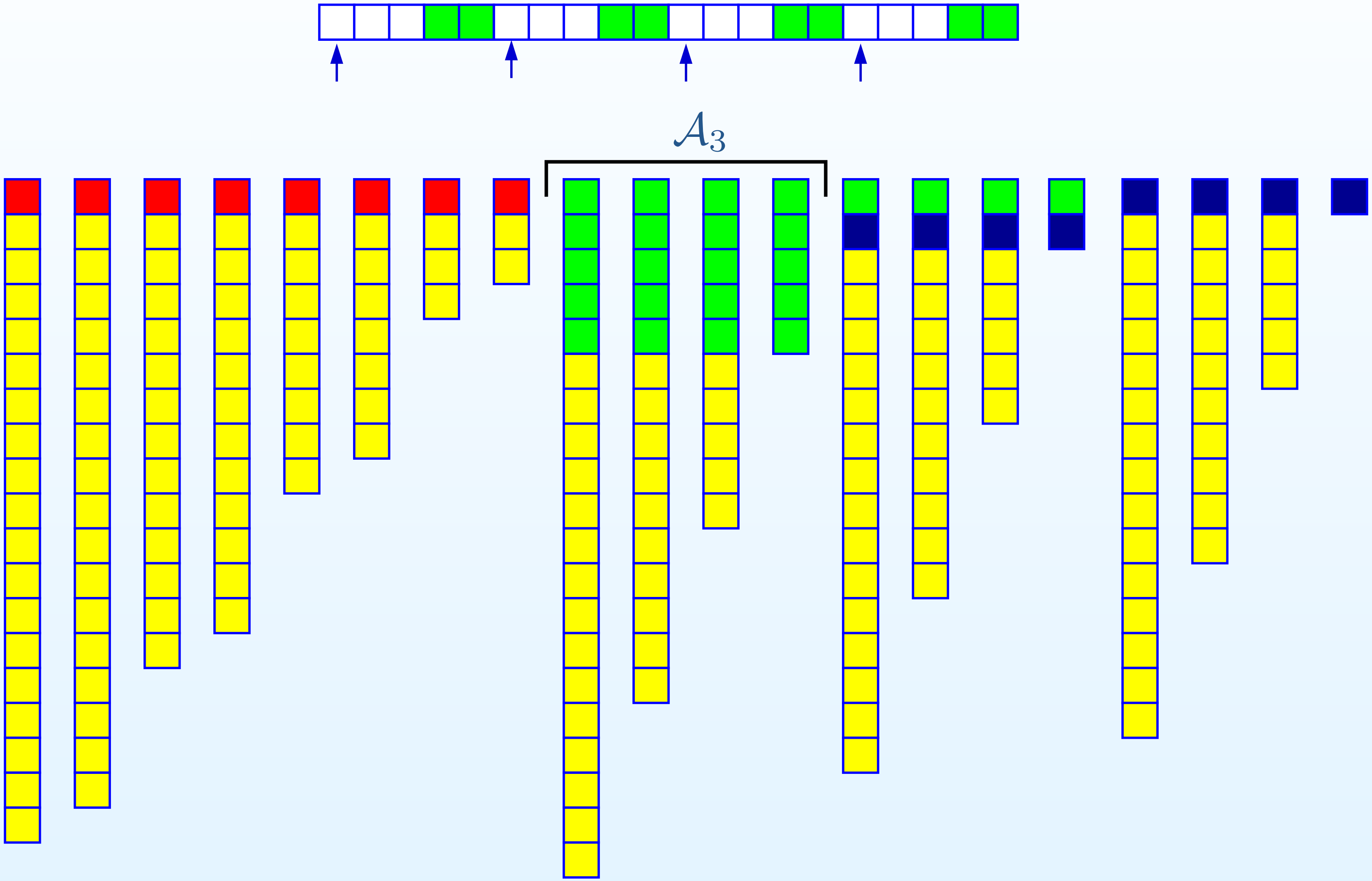

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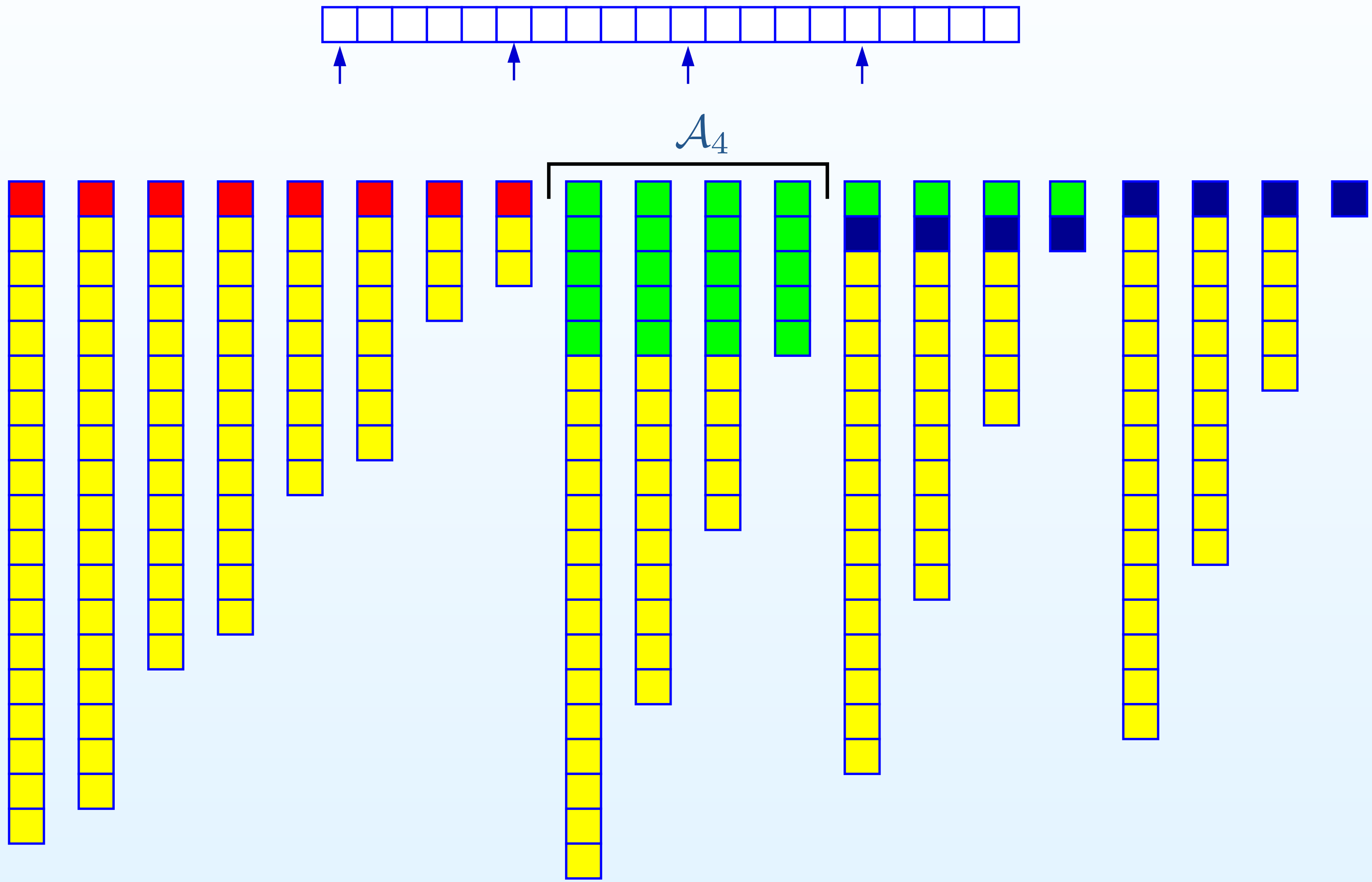

$k = 10$, $l_4 = 8$, Phase 4

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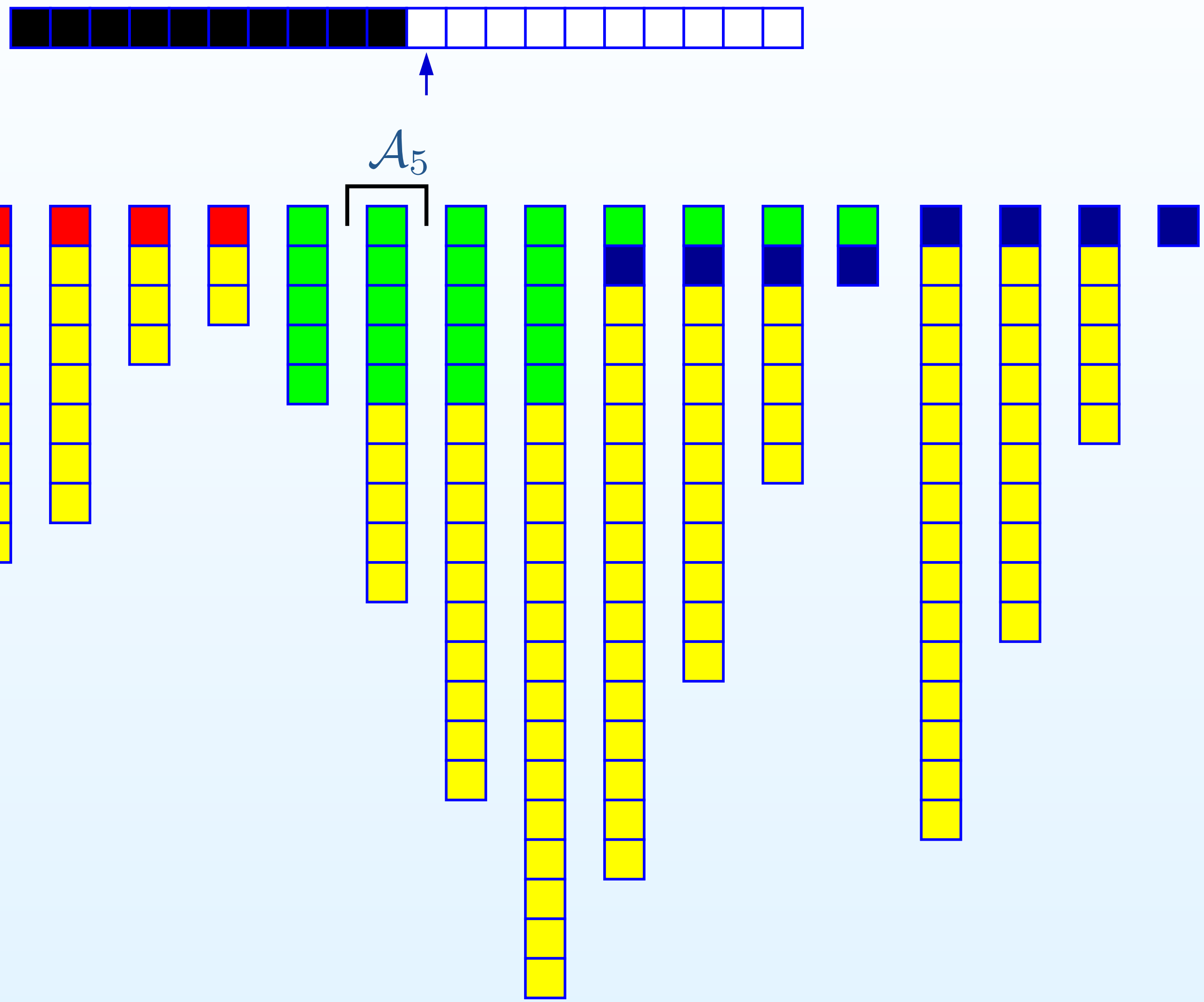
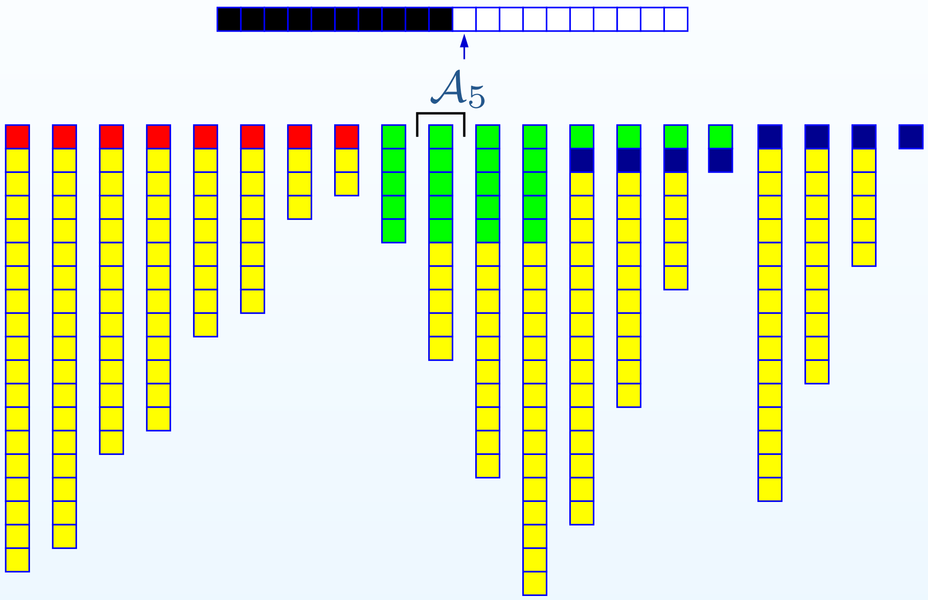


$k = 10$, $l_5 = 9$, Phase 5

Suffix selection, third attempt: reusing inactive suffixes



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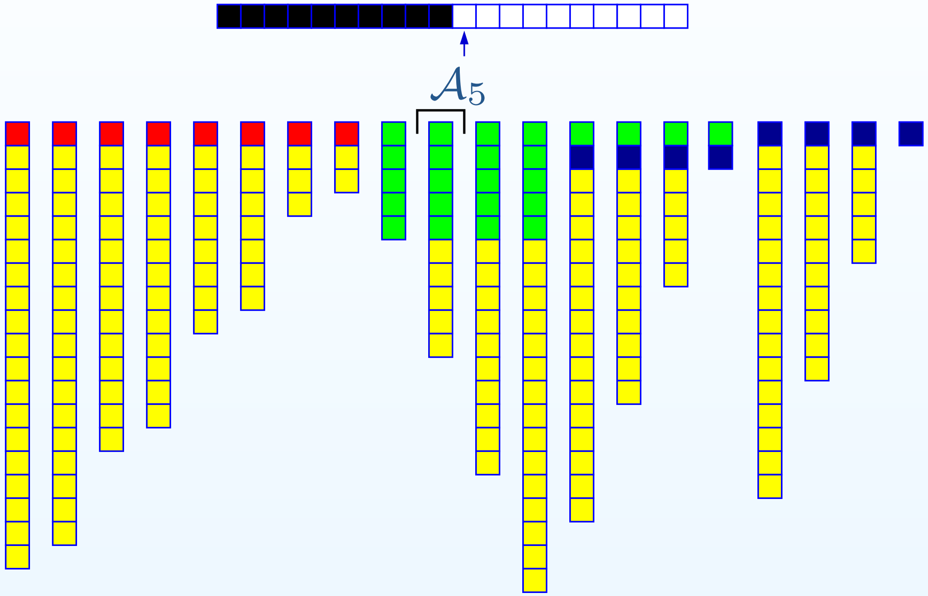

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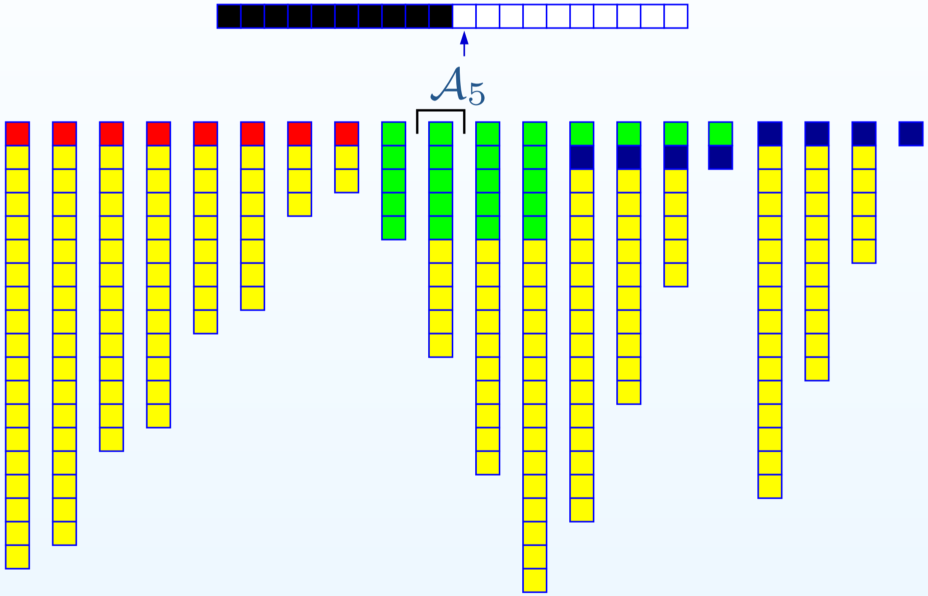

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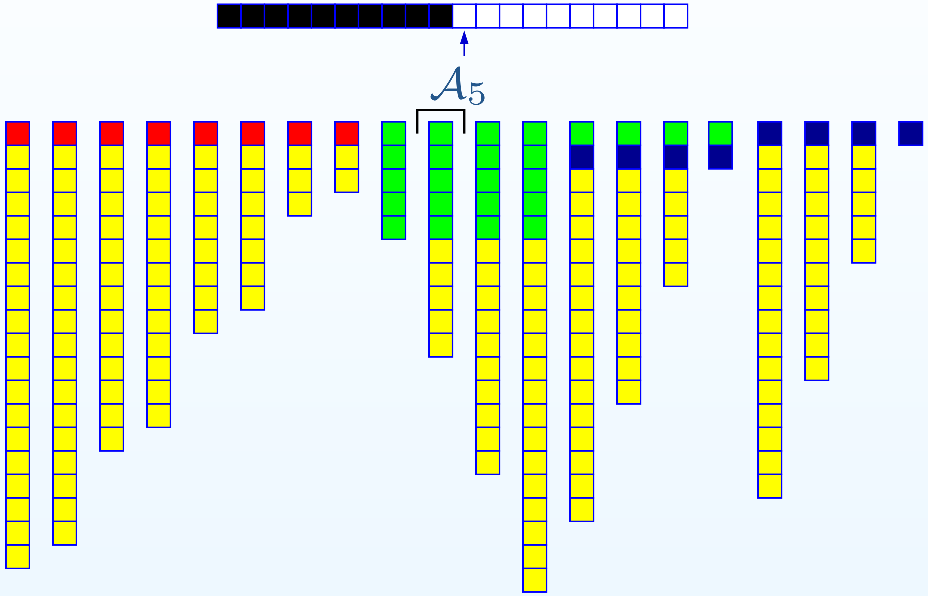

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- ...but any of those accesses to c *can be charged on an active suffix becoming inactive* during the current phase transition.

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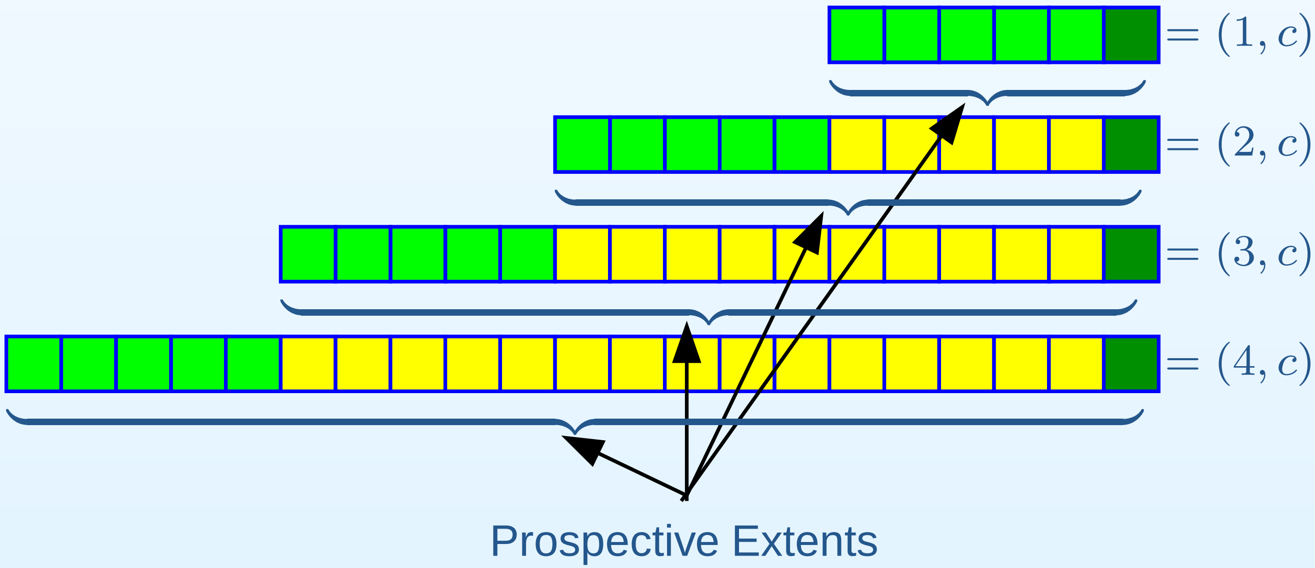

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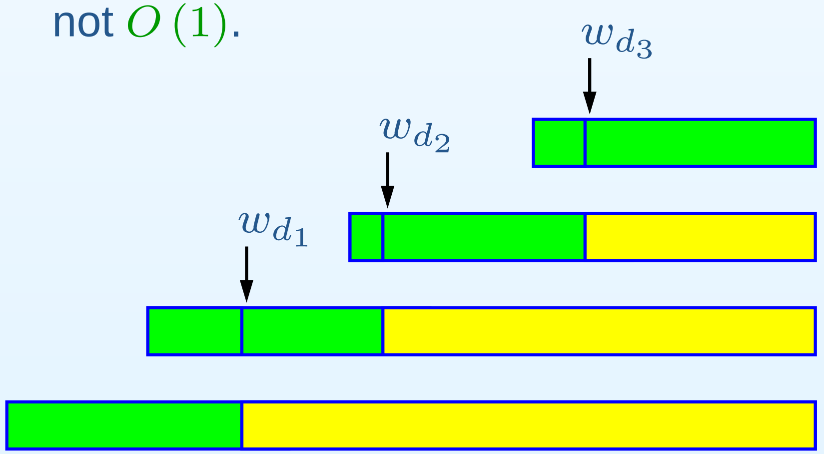

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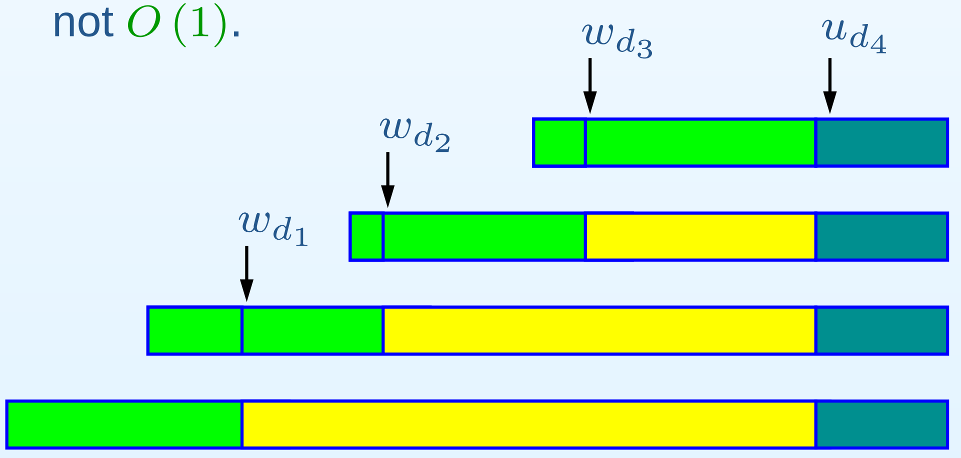

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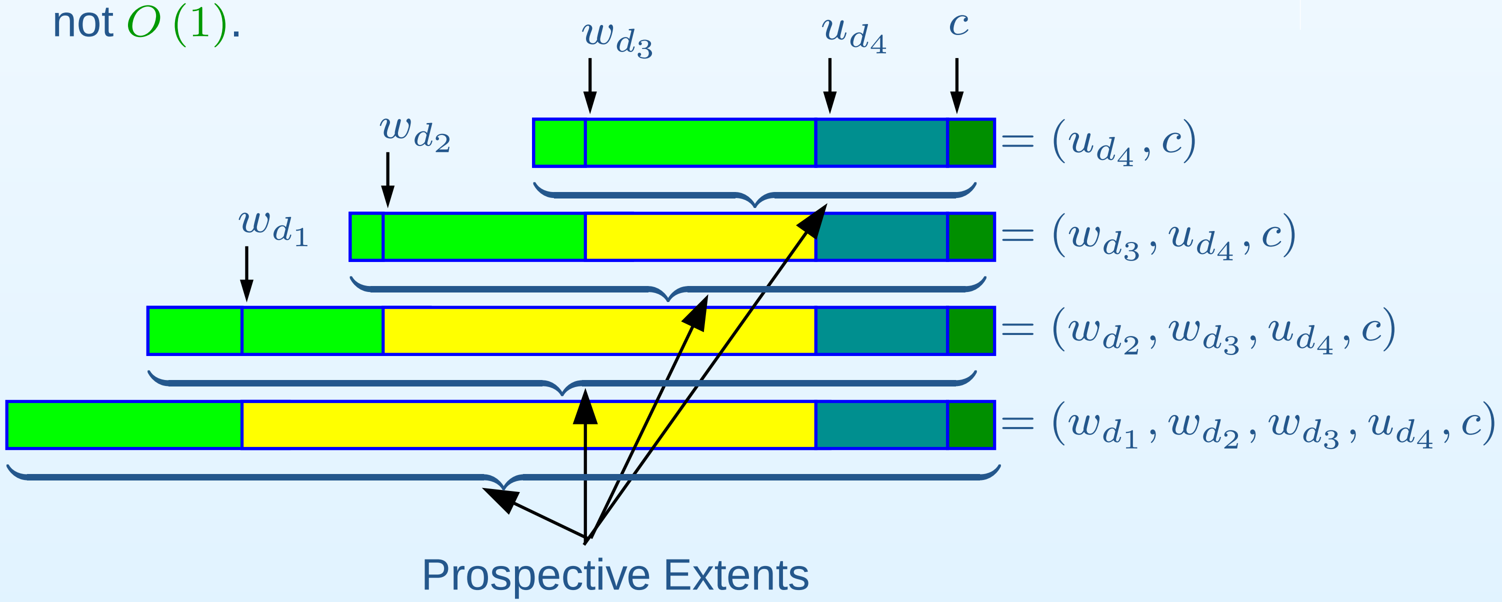

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If we can solve this problem then

The comparison of any two prospective extents of phase t is reduced to one lcp query and one element comparison.

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We can use [Kasai, et al, CPM 2001] and [Harel, Tarjan, SICOMP 13, 1984].

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- We only need to preserve *the length of the longest common prefix* of any two suffixes of G.

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Therefore:

To maintain the forward suffixes, we have to solve a Dynamic Range Maximum Query problem

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- The structure exploits the following crucial fact:

> *Both the integer values stored in the structure and the length of the query intervals* are $O\left(\log^2 n\right)$.

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Therefore,

The total cost for maintaining the forward suffixes during both early and late phases is $O\left(n\right)$.

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- ...*and still aren't*, since *we don't have matching lower bounds* for the new upper bounds.

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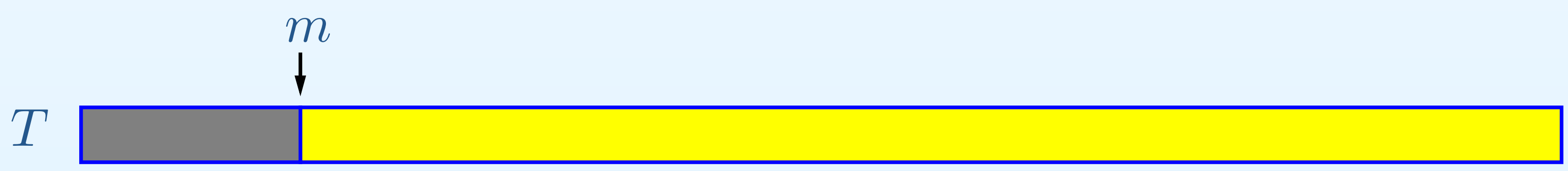

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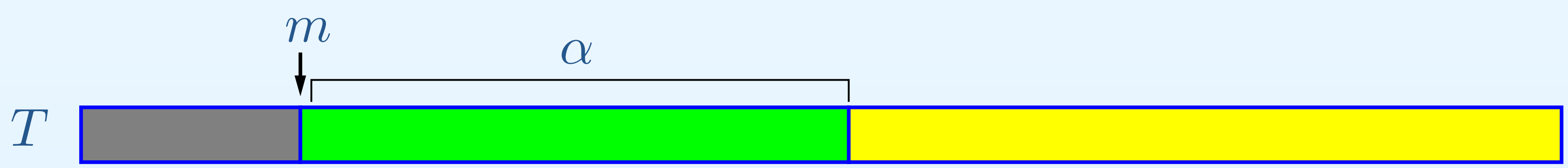

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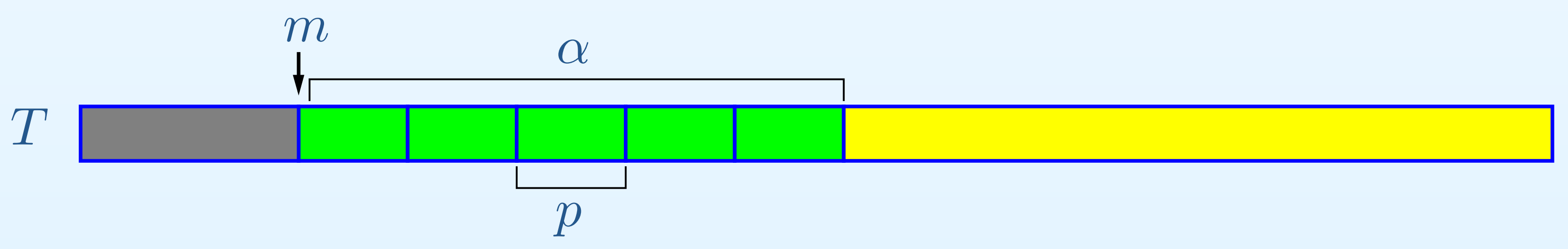

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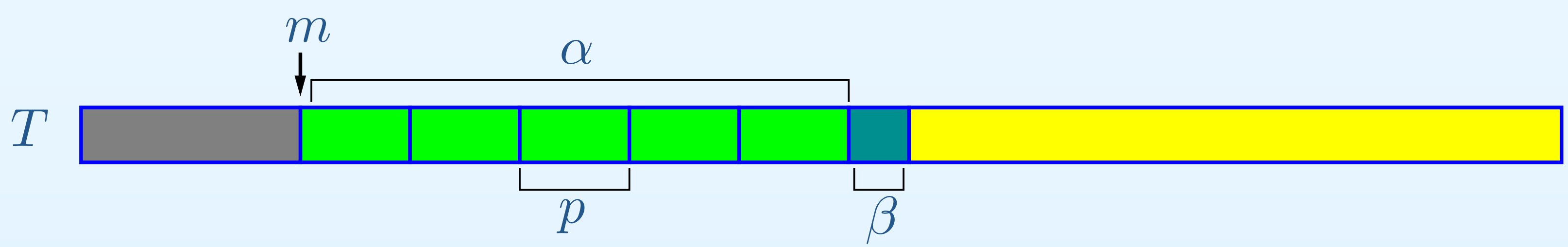

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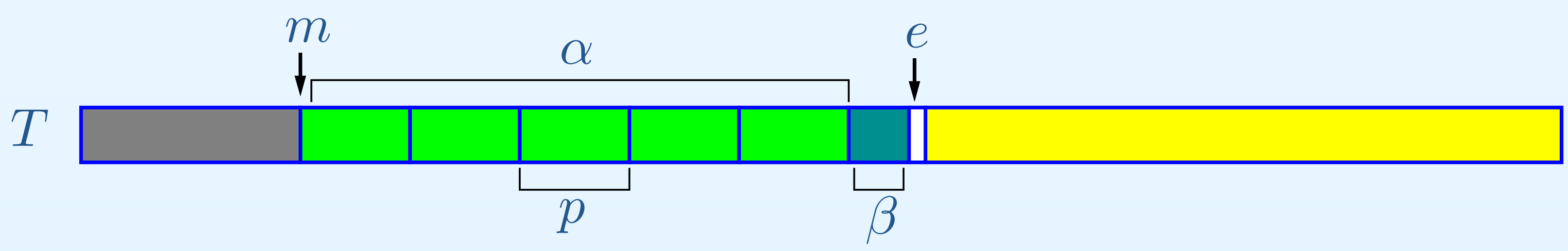

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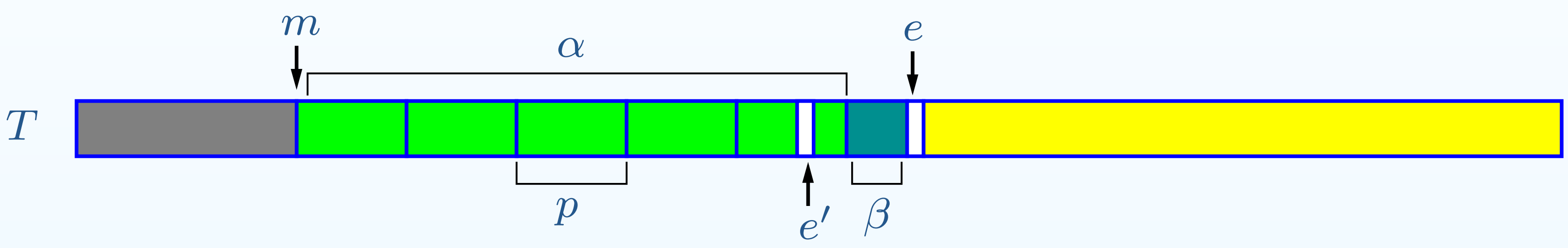

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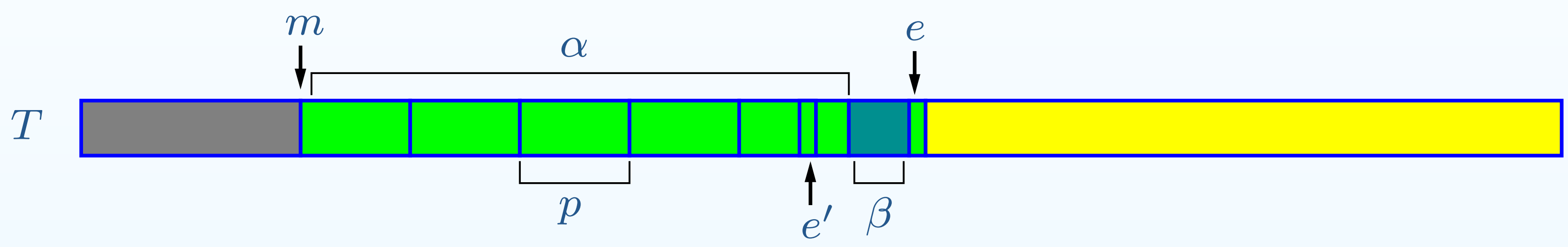

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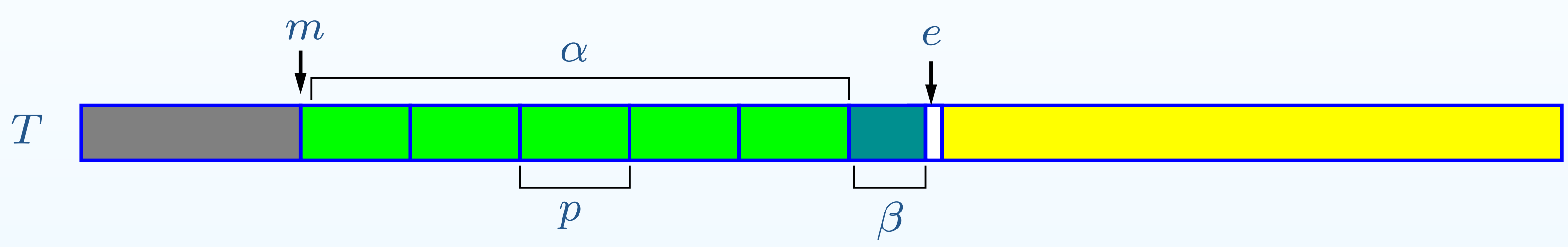

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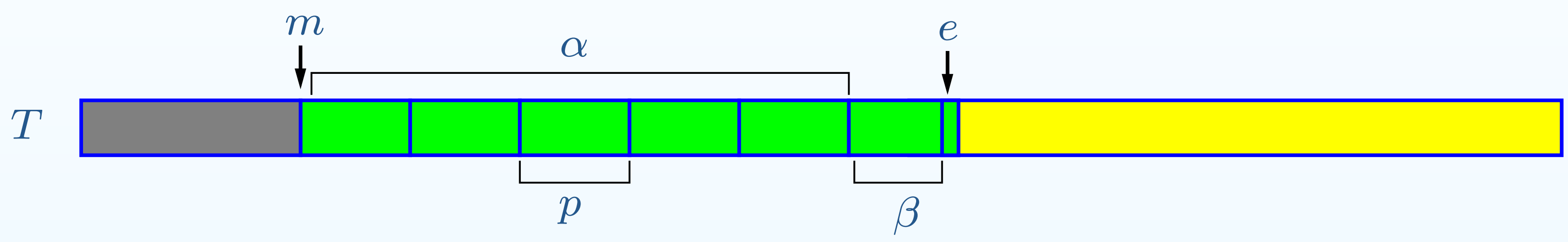

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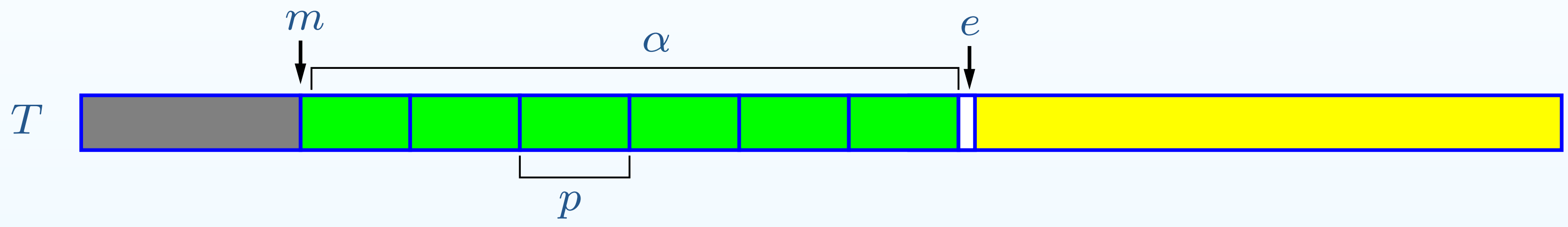

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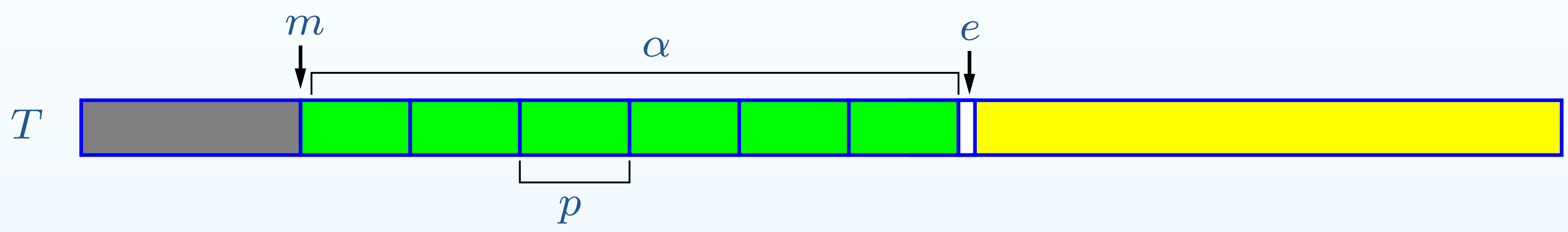

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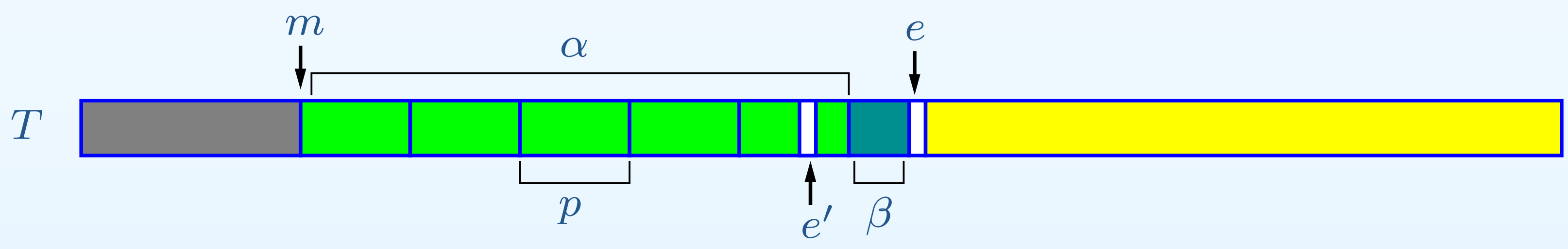

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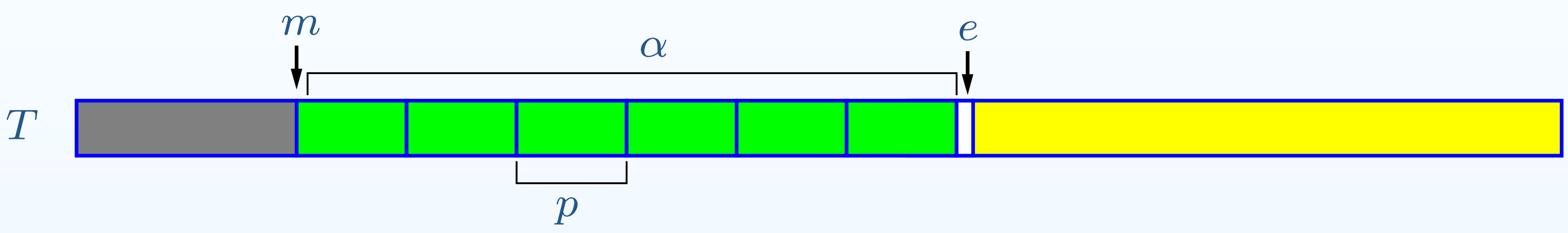

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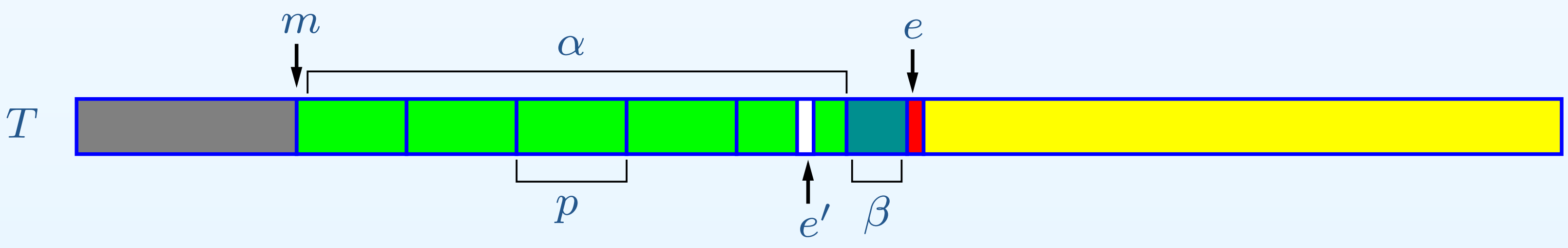

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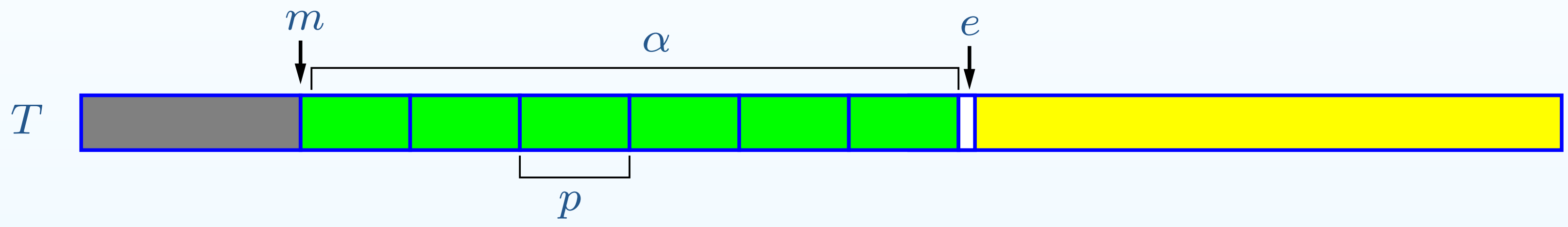

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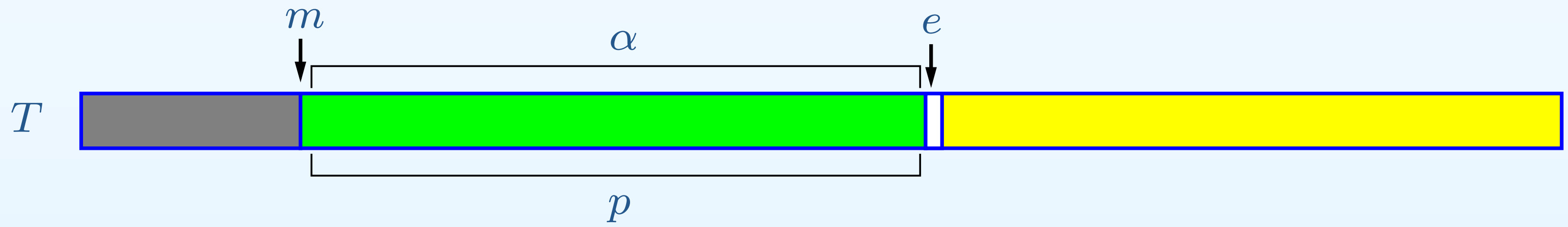

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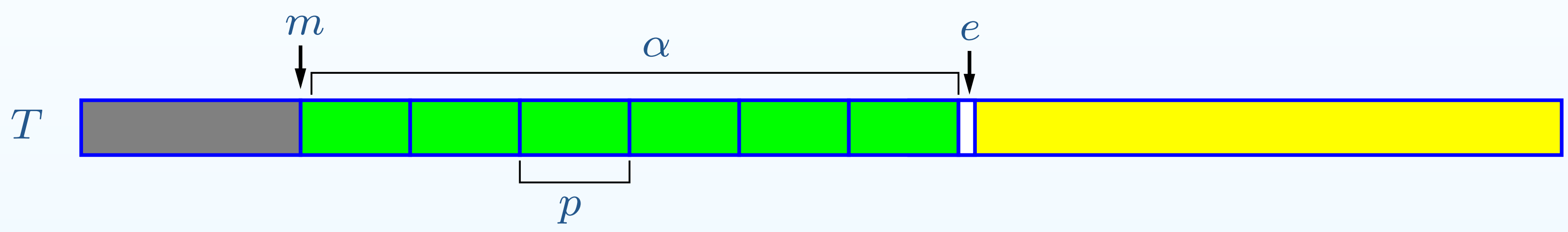

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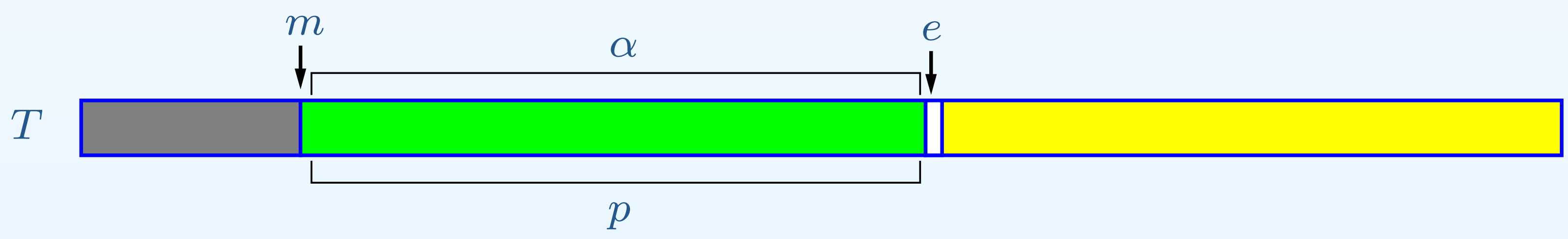

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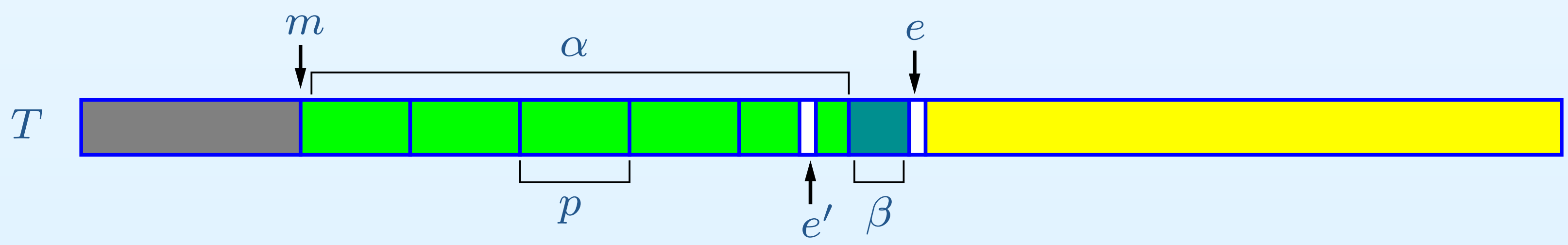

Selection of the Maximum Suffix

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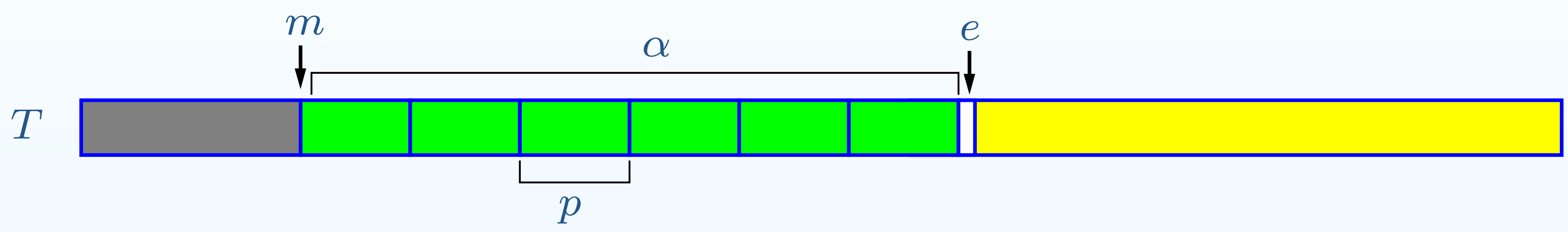

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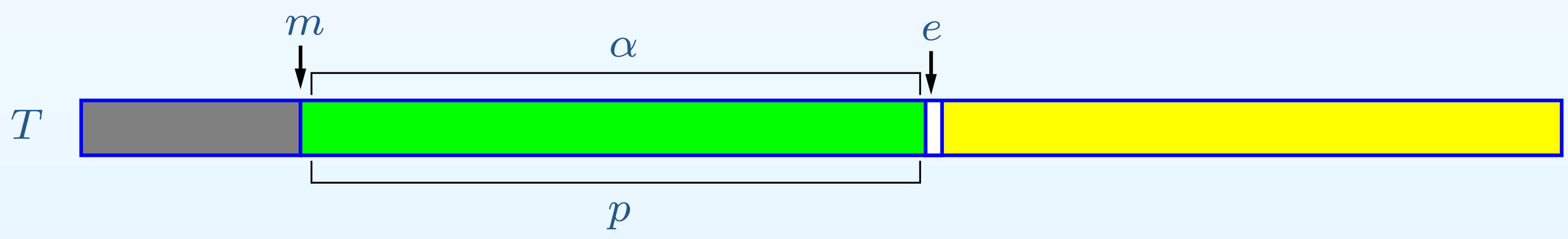

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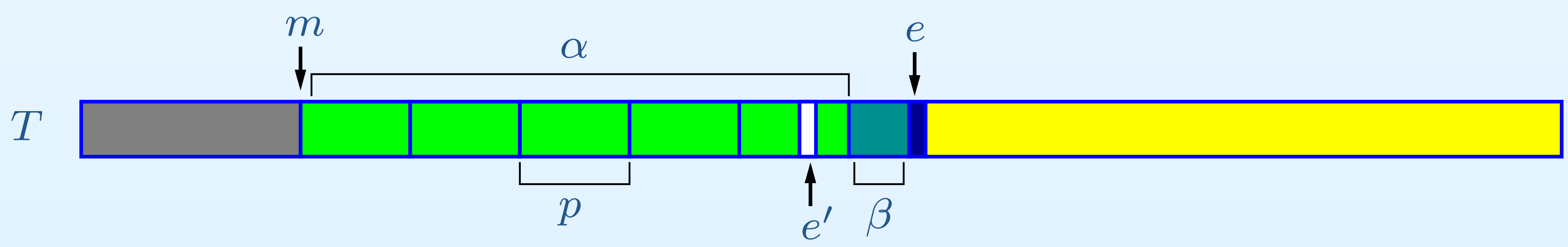

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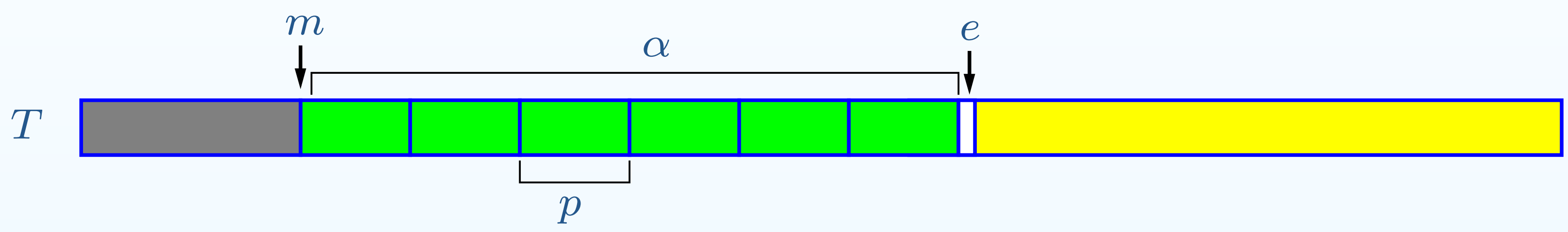

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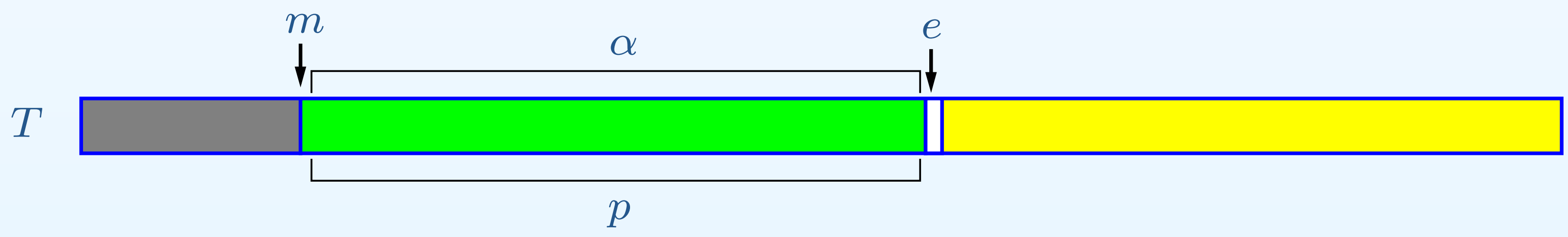

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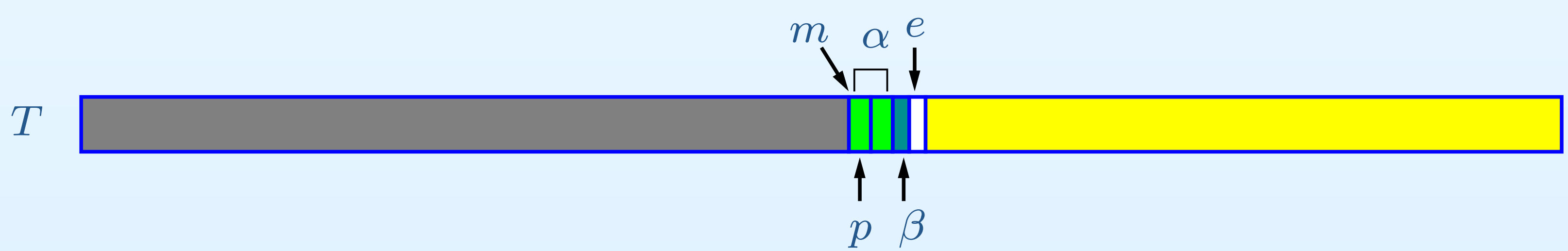

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Worst case scenario for Duval's algorithm:

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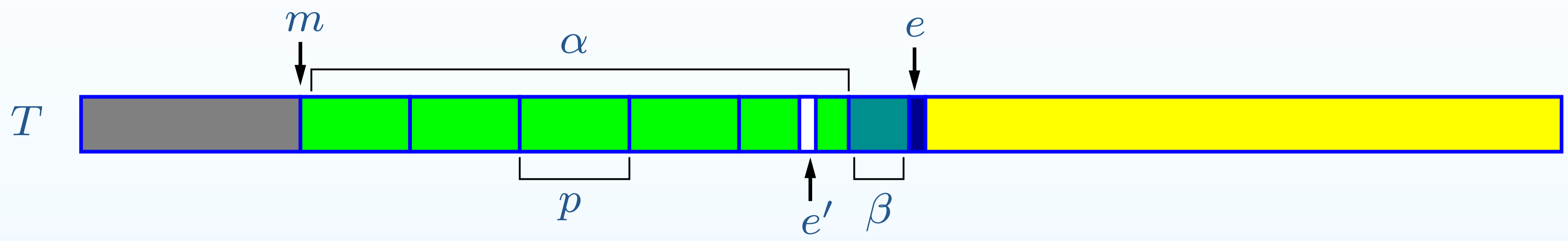

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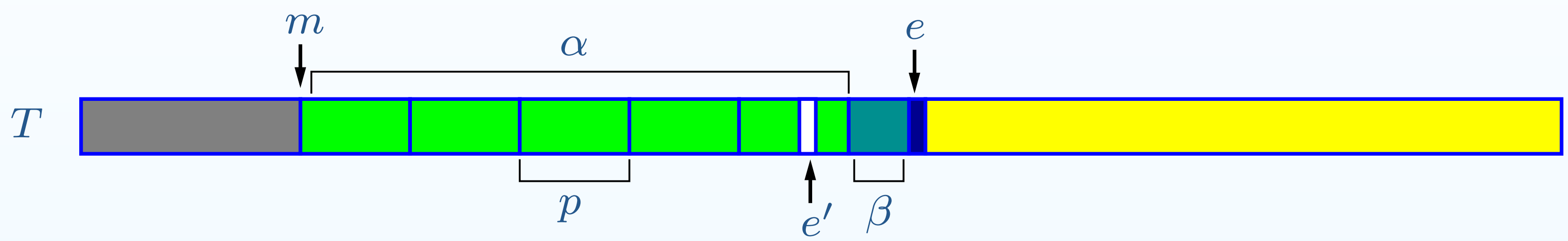

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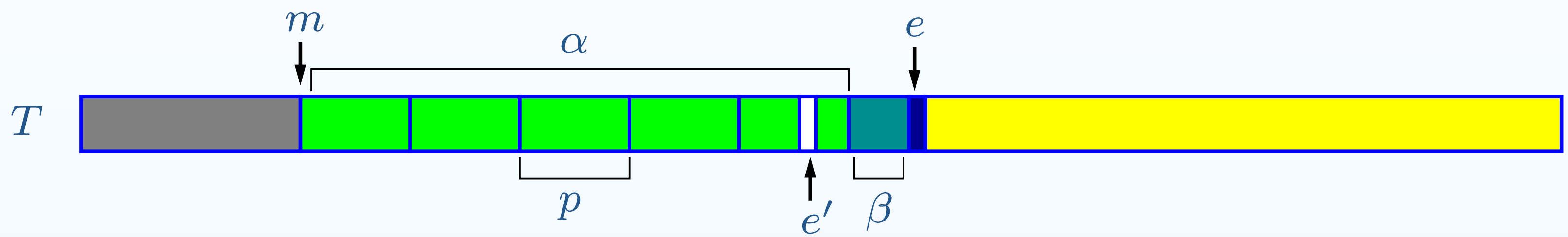

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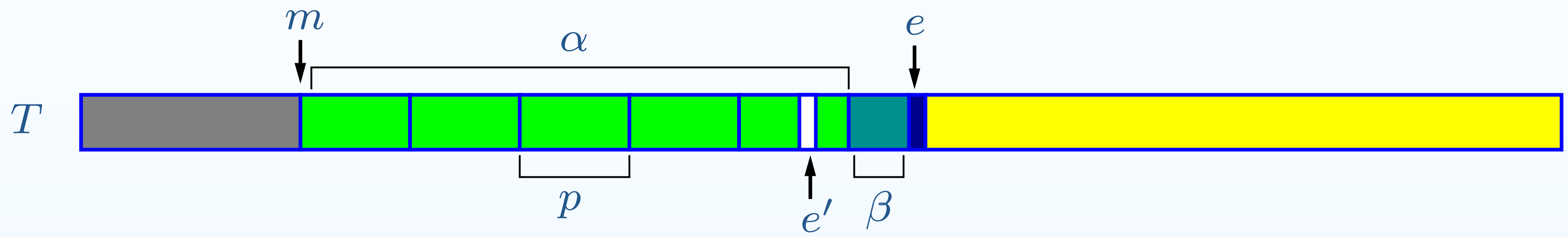

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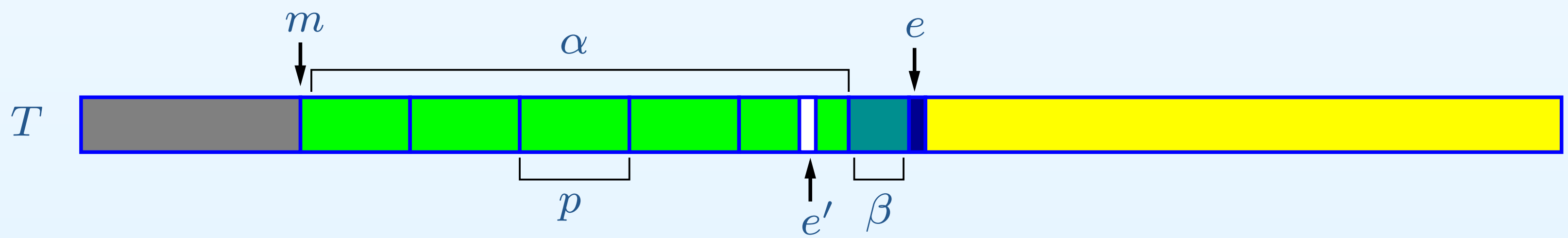

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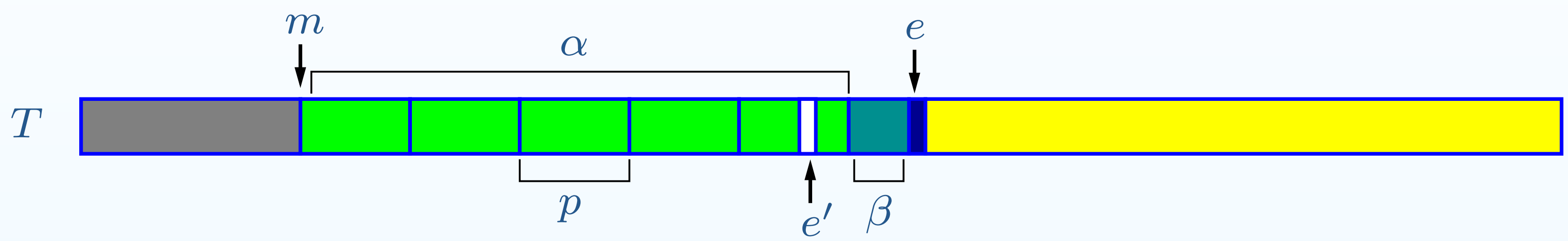

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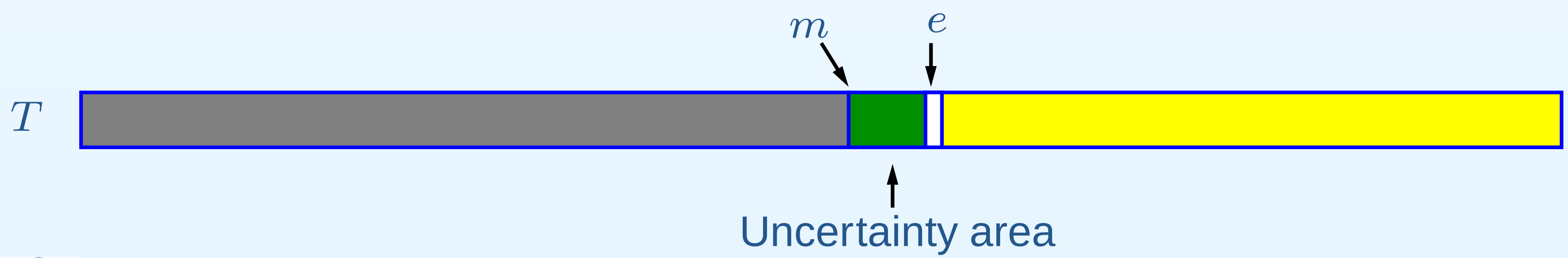

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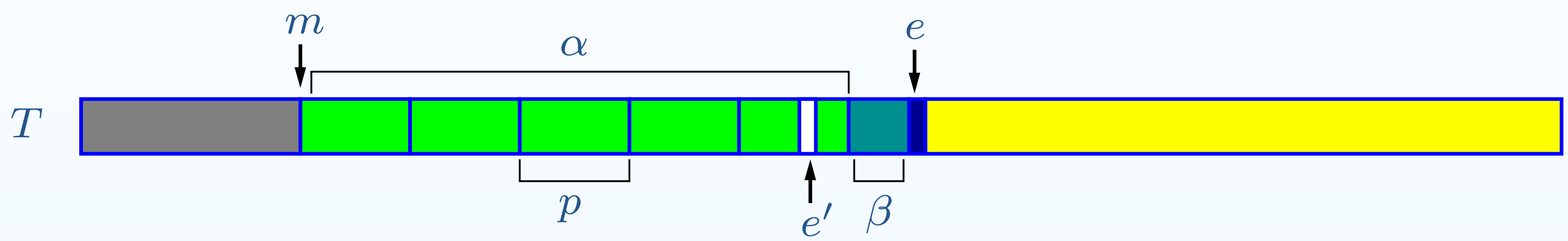

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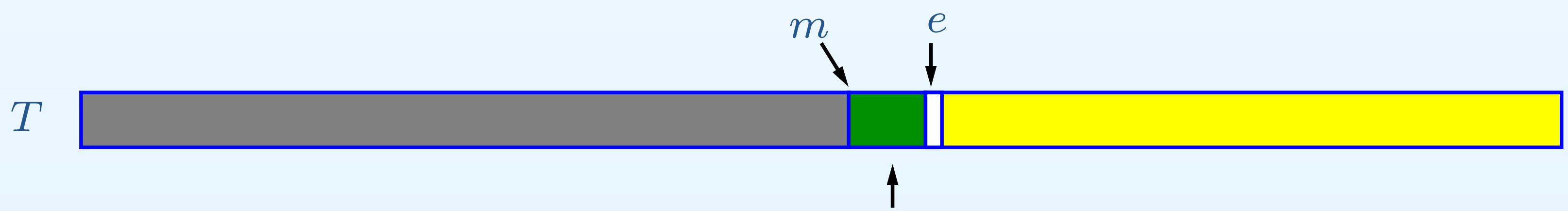

- Obviously, *we still move m* (the current m cannot be the maximum suffix)...
- ...but *we move e too* and we keep *an uncertainty area within which the current maximum suffix starts* (but we don't know where exactly).

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But this is enough to deal with Duval's worst case scenarios

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with less than $\frac{4}{3}n$ comparisons.