Finite Vertex Primitive 2-Path Transitive Graphs

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(joint work with Caiheng Li)

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$\Gamma(\alpha)$=the neighborhood of α, $G_\alpha^{[1]}$=the kernel of G_α acting on $\Gamma(\alpha)$, $G_{\alpha\beta}^{[1]} := G_\alpha^{[1]} \cap G_\beta^{[1]}$.

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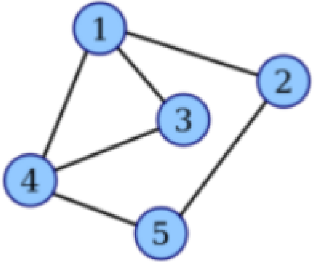
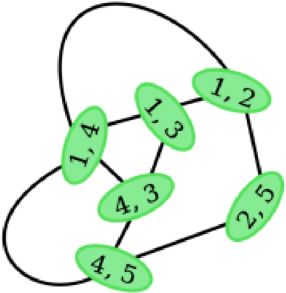
line graph

Γ



$L(\Gamma)$



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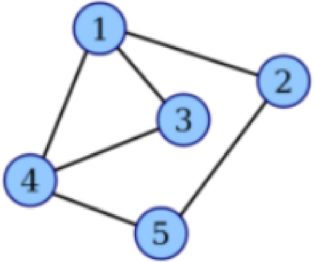

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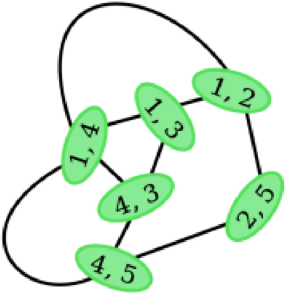

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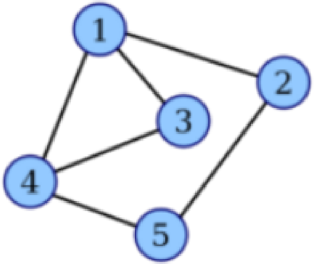

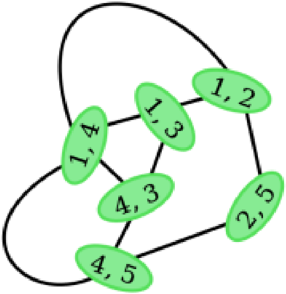

Motivation
Motivations and Aim

Motivations

- motivation 1: To extend the study of symmetrical graphs, **based on**: {2-arc-transitive graphs}⊂{2-path-transitive graphs}⊂{arc-transitive graphs}.
- motivation 2: To construct new half-transitive graphs.

The **aim** is to find a solution for the following problem:

Problem: Classify some special classes of 2-path but not 2-arc transitive graphs (the solution of which will lead to a way of constructing new half-transitive graphs, in our case the specific classes are "vertex-primitive and vertex-biprimitive").

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1. A general study of k-path transitive graphs was first carried out by M. D. E. Conder and C. E. Praeger in 1996.
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The connection between the transitivity of Γ and the transitivity of $L(\Gamma)$ ($G \leq \mathrm{Aut}\Gamma$).

(i) Γ is $(G, 2)$-path-transitive if and only if $L(\Gamma)$ is G-edge transitive;

(ii) Γ is $(G, 2)$-arc-transitive if and only if $L(\Gamma)$ is G-arc transitive;

(iii) Assume that Γ is G-vertex-transitive. Then Γ is $(G, 2)$-path-transitive but not $(G, 2)$-arc-transitive if and only if $L(\Gamma)$ is G-half-transitive.

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A key step of this work is to determine the structure of point stabilizers for 2-path transitive graphs.

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1. If Γ is a (regular) $(G,2)$-path-transitive graph, then it is G-arc-transitive.
2. (Conder and Praeger, 1996) Γ is $(G,2)$-path transitive but not $(G,2)$-arc-transitive if and only if $G_\alpha^{\Gamma(\alpha)}$ is 2-homogeneous but not 2-transitive.
3. (W. M. Kantor, 1969) If G is 2-homogeneous but not 2-transitive of degree n, then $\mathrm{ASL}_1(q) \leq G \leq \mathrm{A\Gamma L}_1(p^e)$, and $n = p^e \equiv 3(\mathrm{mod}\ 4)$ with p prime, e odd.

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Structure of point stabilizers

A result of Weiss: $G_{\alpha\beta}^{[1]} = 1$, is used to proof this theorem:

Structure Theorem

Let Γ be a $(G, 2)$-path transitive but not $(G, 2)$-arc transitive graph, where $\mathrm{G} \le \mathrm{Aut}\Gamma$. Let (α, β) be a arc of Γ. Then either

(1) G_α is faithful on $\Gamma(\alpha)$, and $G_\alpha = G_\alpha^{\Gamma(\alpha)} \le \mathrm{A\Gamma L}_1(q)$, where $q = p^e \equiv 3 (\mathrm{mod}\ 4)$ and p is a prime, or

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(b) $G_\alpha = (G_\alpha^{[1]} \times (\mathbb{Z}_p^e : G_\beta^{[1]})).O$, and $G_{\alpha\beta} = (G_\alpha^{[1]} \times G_\beta^{[1]}).O$, where $O \cong G_{\alpha\beta}^{\Gamma(\alpha)} / (G_\beta^{[1]})^{\Gamma(\alpha)}$;

(c) $q(q-1)/2 \mid |G_\alpha^{\Gamma(\alpha)}|$, and $|G_\alpha| \mid q(\frac{(q-1)e}{2})^2$. In particular, $2 \nmid |G_\alpha|$.

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Primitive type

We focus our attention on the vertex primitive case.

Proposition

Let Γ be a G-vertex primitive, $(G, 2)$-path transitive but not $(G, 2)$-arc transitive graph, where $G \leq \mathrm{Aut}\Gamma$. Then G is affine or almost simple, examples exist for each type.

A general construction Let $H < G$, H a core-free subgroup, g a 2-element. Assume that $g \notin N_G(H)$, $g^2 \in H$, and the action of H on $[H : H \cap H^g]$ by right multiplication is 2-homogeneous but not 2-transitive. Then the graph $\Gamma = \mathrm{Cos}(G, H, HgH)$ is $(G, 2)$-path-transitive but not $(G, 2)$-arc-transitive.

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examples

Example

Let $\Gamma = K_8$, $G = \mathbb{Z}_2^3 : (\mathbb{Z}_7 : \mathbb{Z}_3)$, $G_\alpha = \mathbb{Z}_7 : \mathbb{Z}_3$. Then Γ is vertex-primitive, $(G, 2)$-path transitive but not $(G, 2)$-arc transitive, of affine type.

Example

Let $G = \mathrm{M}$, the Monster simple group. Then G contains a maximal subgroup $H = \mathbb{Z}_{59}{:}\mathbb{Z}_{29} := K : L$. By the ATLAS, the order $|N_G(L)|$ is even, thus there exists a 2-element $g \in N_G(L)$. Furthermore, H acts 2-homogeneously but not 2-transitively on $[H : H \cap H^g]$, so the graph $\Gamma =$Cos(G, H, HgH) is $(G, 2)$-path transitive but not $(G, 2)$-arc transitive, of almost simple type.

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A classification

For AS type, a classification is obtained by using the result of "**Primitive groups with soluble stabilizers**", consists of 7 tables.

G_0	H_0
A_5	$\mathrm{S}_4 \cap G_0$, $(\mathrm{S}_3 \times \mathrm{S}_2) \cap G_0$
A_6	$(\mathrm{S}_4 \times \mathrm{S}_2) \cap G_0$, $(\mathrm{S}_3 \wr \mathrm{S}_2) \cap G_0$, $(\mathrm{S}_2 \wr \mathrm{S}_3) \cap G_0$
A_7	$(\mathrm{S}_4 \times \mathrm{S}_3) \cap G_0$
A_8	$(\mathrm{S}_4 \wr \mathrm{S}_2) \cap G_0$
S_8	$\mathrm{S}_2 \wr \mathrm{S}_4$
A_9	$(\mathrm{S}_3 \wr \mathrm{S}_3) \cap G_0$, $\mathrm{AGL}_2(3) \cap G_0$
A_{12}	$(\mathrm{S}_4 \wr \mathrm{S}_3) \cap G_0$, $(\mathrm{S}_3 \wr \mathrm{S}_4) \cap G_0$
A_{16}	$(\mathrm{S}_4 \wr \mathrm{S}_4) \cap G_0$
A_p	$\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$, $p \neq 7, 11, 17, 23$
S_p	$\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, $p = 7, 11, 17, 23$

Steps of the classification

(a) From the seven tables, find out all maximal subgroups with odd order, we obtained a list (not so long);

(b) From the above list, read off all maximal subgroups with the form $\mathbb{Z}_p^e : L$ or $\mathbb{Z}_p^e \times L : L$, where $L \leq \mathbb{Z}_{(p^e-1)/2} : \mathbb{Z}_e$, we obtained a shorter list of candidates for (G, G_α):

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(c) For each pair (G, H) on the second list, H has the form $H = \mathbb{Z}_p^e : K$. Analyzing $N_G(K)$, since H is maximal and $|K|$ is odd, a $(G, 2)$-path transitive but not $(G, 2)$-arc transitive graph exists if and only if $|N_G(K)|$ is even.

Not all candidates correspond to a 2-path transitive graph.

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By exhausting all the pairs we obtained a complete classification.

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Main result

Theorem

Let Γ be a G-vertex-primitive, $(G, 2)$-path transitive but not $(G, 2)$-arc transitive graph of valency k. Then $k = p^e \equiv 3 (\mathrm{mod}\ 4)$, where p is a prime, and G is affine or almost simple. Furthermore, if G is almost simple, then $\mathrm{soc}(G)$, G_α and k are given in Table A.

Main result
Result table

TABLE A

$\mathrm{soc}(G)$	G_α	k	Conditions	Remark
A_p	$\mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$	p	p prime, $p \equiv 3 (\mathrm{mod}\ 4)$ and $p \neq 7, 11, 23$	
Th	$\mathbb{Z}_{31}{:}\mathbb{Z}_{15}$	31		
B	$\mathbb{Z}_{31}{:}\mathbb{Z}_{15}$	31		
	$\mathbb{Z}_{47}{:}\mathbb{Z}_{23}$	47		
M	$\mathbb{Z}_{59}{:}\mathbb{Z}_{29}$	59		
	$\mathbb{Z}_{71}{:}\mathbb{Z}_{35}$	71		
$\mathrm{PSL}_2(q)$	$\mathbb{Z}_p^e{:}\mathbb{Z}_{(p^e-1)/2}$	q	p prime, $q = p^e \equiv 3 (\mathrm{mod}\ 4)$	$\Gamma = K_{q+1}$

Automorphism groups and Half-transitive graphs

To construct half-transitive graphs, need to determine the automorphism groups of the graphs in Table A, then combining this with a result of Whitney (1932): if $|V\Gamma| \geq 5$, then $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}L(\Gamma)$. we have

Theorem

Let Γ be a graph in Table A. Then the following statements hold:

- For $G = \mathrm{Th}$, B or M, $\mathrm{Aut}(\Gamma) = G$;
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- For $G = \mathrm{A}_p$, $\mathrm{Aut}(\Gamma) = \mathrm{A}_p$ or S_p (depending on g)

In particular, for $G = \mathrm{Th}, \mathrm{B}, M$ or A_p with $\mathrm{Aut}(\Gamma) = \mathrm{A}_\mathrm{p}$, the line graph of Γ is half-transitive.

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What next

1. On vertex biprimitive, 2-path but not 2-arc transitive graphs (recently have been classified by us);
2. On vertex primitive, 2-path but not 2-arc transitive graphs, of affine type (it seems that such graphs are rare);
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Thank you!