TETRAVALENT ONE-REGULAR GRAPHS OF ORDER $4p^2$

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Outline

- Definitions
- Motivation
- Main result
- Essential ingredients in the proof of main result

Definitions

- An **automorphism** of a graph $X = (V, E)$ is an isomorphism of X with itself.

  Thus each **automorphism** α of X is a permutation of the vertex set V which preserves adjacency.

- An **s-arc** in a graph X is an ordered $(s + 1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and also $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$.

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Different types of transitivity

- A graph is **vertex-transitive** if its automorphism group acts transitively on vertices.
- A graph is **edge-transitive** if its automorphism group acts transitively on edges.
- A graph is **arc-transitive** (also called **symmetric**) if its automorphism group acts transitively on arcs.
- If automorphism group acts regularly on the set of s-arcs of X then X is said to be **s-regular**.
- A graph is **one-regular** if its automorphism group acts regularly on the set of its arcs.

Covering graph

A graph $\widetilde{X}$ is called a **covering** of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that

$$p|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$$

is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$.

A **covering** $\widetilde{X}$ of X with a projection p is said to be a **regular** K-**covering** if there is a semiregular subgroup K of $\mathrm{Aut}(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h.

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Tetravalent one-regular graphs

Research in one-regular graphs is interesting from two points of view:

- First, because of the connection to regular maps, in particular to the so-called *chiral maps* (see, for example, [1, 3, 4, 5]). Namely, the underlying graphs of chiral maps admit a one-regular group action with a cyclic vertex stabilizer.
- Second, one may argue that one-regular graphs are interesting in their own right if one is after a description of all arc-transitive graphs of a particular kind. For some classes of Cayley graphs, for example circulants, this has been achieved, with others such as Cayley graphs of dihedral group, all 2-arc-transitive graphs are completely classified in [6] but arc-transitivity is an open problem even for this particular class of graphs.

The current situation

A complete classification of tetravalent one-regular graphs of order p, $pq\,(p \neq q)$ and p^2 is know. In particular, a tetravalent one-regular graph of order p or $pq\,(p \neq q)$ or p^2 is a circulant, due to Cheng, Oxley, Praeger, Wang and Xu. A classification of such graphs can be easily obtained from [9].

Also, Zhou and Feng classified tetravalent one-regular graph of order $2pq$, see[11].

Tetravalent one-regular graphs of order $4p^2$

Our result

A complete classification of tetravalent one-regular graphs of order $4p^2$, p a prime.

Main result

Theorem

Let p be a prime. Then a tetravalent graph X of order $4p^2$ is one-regular if and only if it is isomorphic to one of the graphs listed in the table below. Furthermore, all the graphs listed in this table are pairwise non-isomorphic.

Row	X	$\lvert V(X)\rvert$	$\mathrm{Aut}(X)$
1	$\mathcal{BW}_{12}(5,1,5)$	36	$G_{36} \rtimes \mathbb{Z}_2^2$
2	$\mathcal{GPS}2(4,3,(0\ 1){:}(1\ 2))$	36	$\lvert\mathrm{Aut}(X)\rvert = 144$
3	$\mathcal{NC}^0_{4p^2}$	$4p^2$, $p>7$, $p \equiv \pm 1 \pmod 8$	given in [8, Lemma 8.4]
4	$\mathcal{NC}^1_{4p^2}$	$4p^2$, $p>7$, or $p \equiv 1$ or $3 \pmod 8$	given in [8, Lemma 8.7]
5	$\mathcal{CA}^0_{4p^2}$	$4p^2$, $p \equiv 1 \pmod 4$	$(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$
6	$\mathcal{CA}^1_{4p^2}$	$4p^2$, $p>2$	$(\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$
7	$\mathcal{CN}^2_{4p^2}$	$4p^2$, $p \equiv 1 \pmod 4$	$G^3_{4p^2} \rtimes \mathbb{Z}_4$

Table: Tetravalent one-regular graphs of order $4p^2$.

The essential ingredients in the proof

- The order of the vertex stabilizer of a tetravalent one-regular graph X is 4, and thus its automorphism group is of order $4 \cdot |V(X)|$.

- P. Potočnik and S. Wilson, A census of edge-transitive tetravalent graphs, `http://jan.ucc.nau.edu/ swilson/C4Site/index.html`. This census was used to prove the theorem for $p \leq 3$.

- We use basic group theory, combinatorial techniques, and covering techniques, to show that a Sylow p-subgroup P, $p \geq 5$, of the automorphism group A of a tetravalent one-regular graph of order $4p^2$ is normal in A. And therefore orbits of P give an A-invariant partition.

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Wielandt's theorem

Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. Let $w \in \Omega$. If p^m divides the length of the G-orbit containing w, then p^m also divides the length of the P-orbit containing w.

The quotient graph X/P of a one-regular tetravalent graph X of order $4p^2$ with respect to the orbits of a Sylow p-subgroup P, $p \geq 5$, of its automorphism group A is a cycle of length 4.

- If P is cyclic, then we get graph in row 5.

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Main result

- If P is elementary-abelian then:
  - If P is a minimal normal subgroup of A, then, [8, Theorem 1.2] implies that X is one of the graphs listed in rows 3 and 4.

Row	X	$\|V(X)\|$	$\mathrm{Aut}(X)$
3	$\mathcal{NC}^0_{4p^2}$	$4p^2$, $p > 7$, $p \equiv \pm 1 \pmod 8$	given in [8, Lemma 8.4]
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  - If P is not a minimal normal subgroup of A, then a minimal normal subgroup N of A is isomorphic to $\mathbb{Z}_p$. Considering the quotient graph X_N of X relative to the orbits of N, we have that $|V(X_N)| = 4p$. Then, by a 'reduction' theorem which is deduced from [7, Theorem 1.1], we have that either

    (a) X_N is a cycle of length $4p$, or

    (b) N acts semiregularly on $V(X)$, X_N is a tetravalent connected G/N-arc-transitive graph and X is a regular cover of X_N.

Main result

In case (a), again applying the theorem [8, Theorem 1.2] , we get that X is one of the graphs listed in rows 5 and 6.

Row	X	$\|V(X)\|$	$\mathrm{Aut}(X)$
5	$\mathcal{CA}^0_{4p^2}$	$4p^2$, $p \equiv 1 \pmod 4$	$(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$
6	$\mathcal{CA}^1_{4p^2}$	$4p^2$, $p > 2$	$(\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$

In case (b), X is a covering graph of a symmetric graph of order $4p$. By [10, Theorem 4.1], there are six tetravalent symmetric graphs of order $4p$: $K_{4,4}$, $C_{2p}[2K_1]$, $\mathcal{CA}^0_{4p}$, $\mathcal{CA}^1_{4p}$, $\mathcal{C}(2,p,2)$ and $\mathcal{G}_{28}$.

Then with the use of graph-coverings techniques, we get that X is one of the graphs listed in rows 6 and 7.

Row	X	$\|V(X)\|$	$\mathrm{Aut}(X)$
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Thank you!