Half-edge-transitive graphs and non-normal Cayley graphs

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Outline

1. **Definition and Background**
   - Half-edge-transitive graph
   - Cayley graph
2. **A Sufficient Condition**
3. **Tetravalent non-normal Cayley graphs on alternating group A_6**
4. **References**

Definition

- X: a simple graph (no loops or multiple edges).
- $V(X)$, $E(X)$, $A(X)$: the vertex set, the edge set and the arc set.
- X is vertex-transitive, edge-transitive or arc-transitive(symmetric): $\mathrm{Aut}(X)$ is transitive on $V(X)$, $E(X)$ or $A(X)$.
- X is half-arc-transitive: $\mathrm{Aut}(X)$ is transitive on $V(X)$, $E(X)$, but not on $A(X)$.
- X is half-edge-transitive: A vertex-transitive graph X is half-edge-transitive if X is not edge-transitive and $\mathrm{Aut}(X)$ has two orbits with the same length on the arc set $A(X)$.

Symmetric graphs and half-arc-transitive graphs have been investigated widely.

- It is well known that the lexicographic product $C_n[2K_1]$ is symmetric and $\mathrm{Aut}(C_n[2K_1]) = \mathbb{Z}_2^n \rtimes D_{2n}$, where C_n is the graph of cycle of length n. Thus, tetravalent symmetric graphs can have arbitrary large stabilizers.
- Marušič [24] proved that connected tetravalent half-arc-transitive graphs can have arbitrary large stabilizers.
- In fact, connected tetravalent half-edge-transitive graphs also can have arbitrary large stabilizers.

Cayley graph

G a finite group, $S \subseteq G$, $1 \notin S$, $S = S^{-1} = \{s^{-1} \mid s \in S\}$.

- Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S:
  vertex set $V(\mathrm{Cay}(G, S)) = G$,
  edge set $E(\mathrm{Cay}(G, S)) = \{\{g, sg\} \mid g \in G, s \in S\}$.
- $\mathrm{Cay}(G, S)$ is connected $\Leftrightarrow G = \langle S \rangle$.
- Right regular representation $R(G)$ of G:
  $R(G) = \{R(g) \mid g \in G\}$, where $R(g) : x \mapsto xg, \ \forall x \in G$.
  Clearly, $R(G) \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$.
- A graph X is a Cayley graph on $G \Leftrightarrow \mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices.
- A Cayley graph $\mathrm{Cay}(G, S)$ is said to be normal if $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains $R(G)$ as a normal subgroup.

normal Cayley graph

Wang et al. [32] obtained all disconnected normal Cayley graphs. Let p be a prime.

- p: A Cayley graph of order p is normal if the graph is neither the empty graph nor the complete graph K_p.
- $2p$: Du et al. [9] determined the normality of Cayley graphs of order $2p$.
- p^2: Dobson et al. [8] determined the normality of Cayley graphs of order p^2.

normal Cayley graph on non-abelian simple group

Let $\mathrm{Cay}(G, S)$ be a connected cubic Cayley graph on a non-abelian simple group G.

- Praeger [25] proved that if $N_{\mathrm{Aut}(\mathrm{Cay}(G,S))}(R(G))$ is transitive on edges, then the Cayley graph $\mathrm{Cay}(G, S)$ is normal.
- Fang et al. [11] proved that the vast majority of connected cubic Cayley graphs on non-abelian simple groups are normal.

Fang et al. [12] gave a characterization of a class of tetravalent edge-transitive Cayley graphs of finite non-abelian simple groups in terms of normality.

For more results on the normality of Cayley graphs, we refer the reader to [15, 18, 20, 21, 33, 39].

In view of the above results, one may conclude that most connected Cayley graphs are normal. It is nature to give a question: are there infinite families of connected non-normal Cayley graphs of valency 3 or 4 on non-abelian simple groups????

For valency 4, the answer is yes.

For valency 3???

Theorem

Let $X = \mathrm{Cay}(G, S)$ be a connected tetravalent graph on finite group G. Let $S = \{s_1, s_2, s_3, s_4\}$ and $A = \mathrm{Aut}(X)$. Suppose that there exists an involution h in $G \setminus S$ such that $s_1 \in C_G(h)$, $s_2 = s_1 h$, $s_3^h = s_4$ and $|G : \langle h, s_3, s_4\rangle| = m \geq 2$. Then,

(1) *A contains a subgroup H such that $H \cong \mathbb{Z}_2^m \rtimes R(G)$, and $H_1 \cong \mathbb{Z}_2^{m-1} \rtimes \mathbb{Z}_2$;*

(2) *X is non-normal, and X is symmetric or half-edge-transitive;*

(3) *If $\langle h, s_3, s_4\rangle \not\cong D_8$ then X is half-edge-transitive.*

(4) *If m is a prime then X is half-edge-transitive except for $X \cong Q_4$, the 4-dimensional hypercube, or $X \cong \mathrm{Cay}(S_4, \{(1\ 2),\ (3\ 4),\ (1\ 3),\ (2\ 4)\})$.*

Example 1

Let $p \equiv 1 \pmod 4$ be a prime and let A_p be the alternating group on $\Omega = \{1, 2, \cdots, p\}$. Take

$$
\begin{aligned}
s_1 &= (1\ 2)\cdots(\tfrac{p-3}{2}\ \tfrac{p-1}{2})(\tfrac{p+1}{2})(\tfrac{p+3}{2}\ \tfrac{p+5}{2})\cdots(p-1\ p),\\
s_2 &= (1\ p-1)(2\ p)(3\ p-3)(4\ p-2)\cdots(\tfrac{p-1}{2}\ \tfrac{p+5}{2})(\tfrac{p+1}{2}),\\
s_3^1 &= (1\ 2)(3\ p)(4\ p-1)\cdots(\tfrac{p-1}{2}\ \tfrac{p+7}{2})(\tfrac{p+1}{2}\ \tfrac{p+5}{2})(\tfrac{p+3}{2}),\\
s_4^1 &= (p\ p-1)(p-2\ 1)(p-3\ 2),\cdots(\tfrac{p+3}{2}\ \tfrac{p-5}{2})(\tfrac{p+1}{2}\ \tfrac{p-3}{2})(\tfrac{p-1}{2}),\\
s_3^2 &= (1\ p)(2\ 3)\cdots(\tfrac{p-1}{2}\ \tfrac{p+1}{2})(\tfrac{p+3}{2})(\tfrac{p+5}{2}\ \tfrac{p+7}{2})\cdots(p-2\ p-1),\\
s_4^2 &= (1\ p)(2\ 3)\cdots(\tfrac{p-5}{2}\ \tfrac{p-3}{2})(\tfrac{p-1}{2})(\tfrac{p+1}{2}\ \tfrac{p+3}{2})\cdots(p-2\ p-1),\\
s_3^3 &= (1\ 3\ 5\ \cdots\ p-2\ p),\\
s_4^3 &= (1\ p\ p-2\ \cdots\ 5\ 3),\\
s_3^4 &= (1\ 2\ 3\ \cdots\ p-1\ p),\\
s_4^4 &= (1\ p\ p-1\ \cdots\ 3\ 2).
\end{aligned}
$$

Set $S_i = \{s_1, s_2, s_3^i, s_4^i\}$ for $i = 1,\ 2,\ 3,\ 4$. Then $X_i = \mathrm{Cay}(A_p, S_i)$ is half-edge-transitive and non-normal.

$p = 5$

Let $p = 5$, $a = (1\ 3)(2\ 4)$, $b = (1\ 4)(2\ 3)$, $c = (1\ 2)(4\ 5)$ and $d = (1\ 4)(2\ 5)$. Then

$$\begin{aligned} S_1 &= \{c, d, (1\ 2)(3\ 5), (1\ 3)(4\ 5)\} \cong \{a, b, (1\ 3)(2\ 5), (1\ 5)(2\ 4)\}, \\ S_2 &= \{c, d, (1\ 5)(2\ 3), (1\ 5)(3\ 4)\} \cong \{a, b, (1\ 2)(3\ 5), (1\ 2)(4\ 5)\}, \\ S_3 &= \{c, d, (1\ 3\ 5), (1\ 5\ 3)\} \cong \{a, b, (1\ 2\ 5), (1\ 5\ 2)\}, \\ S_4 &= \{c, d, (1\ 2\ 3\ 4\ 5), (1\ 5\ 4\ 3\ 2)\} \cong \{a, b, (1\ 3\ 5\ 4\ 2), (1\ 2\ 4\ 5\ 3)\end{aligned}$$

From [40], up to isomorphic, there are only four tetravalent non-normal Cayley graphs on alternating group A_5, corresponding the four graphs above. Further, $2^6 \mid |\mathrm{Aut}(\mathrm{Cay}(A_5, S_1))_1| = |\mathrm{Aut}(\mathrm{Cay}(A_5, S_4))_1|$, and $2^{10} \mid |\mathrm{Aut}(\mathrm{Cay}(A_5, S_2))_1| = |\mathrm{Aut}(\mathrm{Cay}(A_5, S_3))_1|$. By Magma [4], $|\mathrm{Aut}(\mathrm{Cay}(A_5, S_1))| = |\mathrm{Aut}(\mathrm{Cay}(A_5, S_4))| = 2^6 \cdot 60$ and $|\mathrm{Aut}(\mathrm{Cay}(A_5, S_2))| = |\mathrm{Aut}(\mathrm{Cay}(A_5, S_3))| = 2^{11} \cdot 60$, it means that the lower bound is sharp.

normal Cayley graphs on alternating group A_5

- Xu and Xu [40] determined the normality of connected Cayley graphs of valency 3 and 4 on alternating group A_5.
- Zhou and Feng [41] determined the normality of connected Cayley graphs of valency 5 on alternating group A_5. Furthermore, they gave two sufficient conditions for non-normal Cayley graphs, and constructed some infinite families of non-normal Cayley graphs of valency 5.

Example 2

Let $p = 8k + 7$ be a prime, k a positive integer with $k \neq 1 \pmod 3$ and let A_p be the alternating group on $\Omega = \{1, 2, \cdots, p\}$. Take $s_1 = (1\ 3\ 2\ 4)\cdots(p-6\ p-4\ p-5\ p-3)(p-2\ p-1)(p)$, $s_3 = (1\ 2)(3\ 4\ 5\ 6)\cdots(p-8\ p-7\ p-6\ p-5)(p-4\ p-2\ p-3\ p)(p-1)$, $s_2 = s_1^{-1}$ and $s_4 = s_3^{-1}$.

Then $X = \mathrm{Cay}(A_p, \{s_1, s_2, s_3, s_4\})$ is half-edge-transitive and non-normal.

In this case, $D \cong D_8$ and $m = \frac{p!}{16}$.

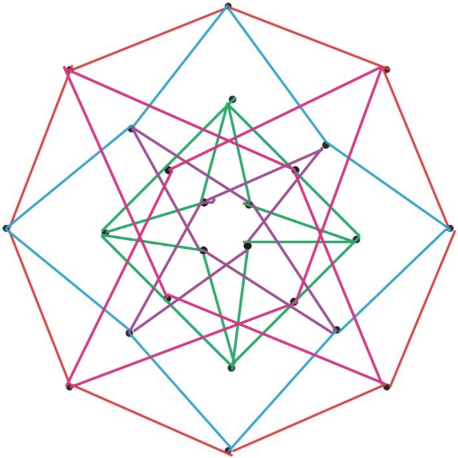
Example 3

Let $n = 8k + 6$ be an integer with $k \neq 1 \pmod 3$ and let A_n be the alternating group on $\Omega = \{1, 2, \cdots, n\}$. Take

$$\begin{aligned}
s_1 &= (1\ \tfrac{n+2}{2})(2\ 3\ n\ n-1)(4\ 5\ n-2\ n-3)\cdots(\tfrac{n-2}{2}\ \tfrac{n}{2}\ \tfrac{n+6}{2}\ \tfrac{n+4}{2}),\\
s_2 &= (1\ \tfrac{n+2}{2})(2\ n-1\ n\ 3)(4\ n-3\ n-2\ 5)\cdots(\tfrac{n-2}{2}\ \tfrac{n+4}{2}\ \tfrac{n+6}{2}\ \tfrac{n}{2}),\\
s_3^1 &= (2\ 3\ 4\ \cdots\ n),\\
s_4^1 &= (2\ n\ \cdots\ 4\ 3),\\
s_3^2 &= (1\ \tfrac{n+6}{2})(2\ 3)(4\ 5)\cdots(\tfrac{n-2}{2}\ \tfrac{n}{2})(\tfrac{n+4}{2}\ \tfrac{n+10}{2})(\tfrac{n+8}{2}\ \tfrac{n+14}{2})\cdots(n-3\\
s_4^2 &= (1\ \tfrac{n-2}{2})(2\ 5)(4\ 7)\cdots(\tfrac{n-6}{2}\ \tfrac{n}{2})(\tfrac{n+4}{2}\ \tfrac{n+6}{2})(\tfrac{n+8}{2}\ \tfrac{n+10}{2})\cdots(n-1\ n
\end{aligned}$$

Set $S_i = \{s_1, s_2, s_3^i, s_4^i\}$ for $i = 1,\ 2$. Then $X_i = \mathrm{Cay}(A_n, S_i)$ is half-edge-transitive and non-normal.

$\mathrm{Cay}(S_4, \{(1\ 2),\ (3\ 4),\ (1\ 3),\ (2\ 4)\})$



Corollary

- Tetravalent half-edge-transitive graphs can have arbitrary large stabilizer.
- There exist infinite families of tetravalent non-normal Cayley graphs on non-abelian simple groups.

Ideas for the proof

Set $D = \langle h, s_3, s_4 \rangle$. Then $D \cong D_{2n}$ for some integer $n > 2$. Note that $s_1 \in C_G(h)$, $s_2 = s_1 h$, $s_3^h = s_4$ and $|G : \langle h, s_3, s_4 \rangle| = m \geq 2$.

- Fact 1: $o(s_1) = 2$ or $o(s_1) = 4$ and $h = s_1^2$; $o(s_3) = 2$ or $o(s_3) > 2$ and $s_4 = s_3^{-1}$.
- Fact 2: $D \lhd G \Leftrightarrow m = 2$.

(1) $\mathbb{Z}_2^m \rtimes R(G) \leq \mathrm{Aut}(X)$

Let Dg_i, $i = 0, 1, \cdots, m-1$, be all cosets of D in G such that $g_0 = 1$ and $g_1 = s_1$. For a given coset Dg_i, define the permutation α_i on G by

$$x^{\alpha_i} = \begin{cases} hx & x \in Dg_i, \\ x & x \notin Dg_i. \end{cases} \quad (1)$$

Then $\alpha_i \in \mathrm{Aut}(X)$. Set $T = \langle \alpha_0, \alpha_1, \cdots, \alpha_{m-1} \rangle$. Then T is normalized by $R(G)$ and $T \cong \mathbb{Z}_2^m$. Thus, $H = T \rtimes R(G) \cong \mathbb{Z}_2^m \rtimes R(G) \leq \mathrm{Aut}(X)$ and $H_1 \cong \mathbb{Z}_2^{m-1} \rtimes \mathbb{Z}_2$.

By (1), the order of A_1 has a lower bound 2^m.

(2) **non-normal**

- $m > 2$: Then α_2 fixes $\{s_1, s_2, s_3, s_4\}$ pointwise. Thus $\alpha_2 \in A_1$, but $\alpha_2 \notin \mathrm{Aut}(G, S)$.
- $m = 2$: Then $D \lhd G$. Thus $\alpha_1 \in A_1$, but $\alpha_1 \notin \mathrm{Aut}(G, S)$.

Thus, X is non-normal.

(3) symmetric or half-edge-transitive

$o(s_1) = 2$: $(1, s_1)^{R(s_1)} = (s_1, 1)$, $(s_1, 1)^{R(s_1)} = (1, s_1)$;
$o(s_1) = 4$: $(1, s_1)^{\alpha_1 R(s_1)} = (s_1, 1)$, $(s_1, 1)^{\alpha_1 R(s_1)} = (1, s_1)$.

$o(s_3) = 2$: $(1, s_3)^{R(s_3)} = (s_3, 1)$, $(s_3, 1)^{R(s_3)} = (1, s_3)$;
$o(s_3) > 2$: $(1, s_3)^{\alpha_0 R(s_3)} = (s_3, 1)$, $(s_3, 1)^{\alpha_0 R(s_3)} = (1, s_3)$.

Thus, A has at most two orbits on the arc set of X and if it has two orbits, they have the same length.

It follows that X is symmetric or half-edge-transitive.

(4) if $D \ncong D_8$ then X is half-edge-transitive

$D \ncong D_8$.

Then either there is one 4-cycle passing through the edge $\{1, s_1\}$ and no 4-cycle passing through the edge $\{1, s_3\}$, or there are three 4-cycles passing through the edge $\{1, s_1\}$ and two 4-cycles passing through the edge $\{1, s_3\}$.

Thus, X is half-edge-transitive.

$D \cong D_8$ and m is a prime.

- $m = 2$: Then $G = D_8 \times \mathbb{Z}_2 = \langle a, b, c | a^4 = b^2 = c^2 = [a, c] = [b, c] = 1, a^b = a^{-1} \rangle$ and $S = \{a, a^{-1}, c, bc\}$ or $\{ab, a^{-1}b, c, bc\}$. Thus, $X \cong Q_4$(the 4-dimensional hypercube) is symmetric.
- $m \geq 3$: Then $G = S_4$ and $S = \{(1\ 2),\ (3\ 4),\ (1\ 2\ 3\ 4),\ (1\ 4\ 3\ 2)\}$ or $\{(1\ 2),\ (3\ 4),\ (1\ 3),\ (2\ 4)\}$, or $\{(1\ 3\ 2\ 4),\ (1\ 4\ 2\ 3),\ (1\ 2\ 3\ 4),\ (1\ 4\ 3\ 2)\}$. Thus, $X \cong \mathrm{Cay}(S_4, \{(1\ 2),\ (3\ 4),\ (1\ 3),\ (2\ 4)\})$ is symmetric.

Tetravalent non-normal Cayley graphs on A_6

Theorem

Let $\mathrm{Cay}(A_6, S)$ *be a connected tetravalent non-normal Cayley graph on alternating group* A_6*. Then* $S \cong \{(1\ 4)(2\ 3\ 6\ 5), (1\ 4)(2\ 5\ 6\ 3), (2\ 3\ 4\ 5\ 6), (2\ 6\ 5\ 4\ 3)\}$ *or* $\{(1\ 4)(2\ 3\ 6\ 5), (1\ 4)(2\ 5\ 6\ 3), (1\ 2)(5\ 6), (1\ 6)(2\ 3)\}$.

subdegrees of $\mathrm{PSU}(4,2)$

Lemma

Let $m > 1$ be a positive integer dividing 180, and let $\mathrm{PSU}(4,2)$ has a transitive action of degree m. Then $m = 40,\ 45,\ 90,$ or 120, Further,

(1) *if $m = 40$, then the subdegrees of $\mathrm{PSU}(4,2)$ are $1,\ 12,\ 27$;*

(2) *if $m = 45$, then the subdegrees of $\mathrm{PSU}(4,2)$ are $1,\ 12,\ 32$;*

(3) *if $m = 90$, then the subdegrees of $\mathrm{PSU}(4,2)$ are $1,\ 1,\ 24,\ 32,\ 32$;*

(4) *if $m = 120$, then the subdegrees of $\mathrm{PSU}(4,2)$ are $1,\ 1,\ 1,\ 27,\ 27,\ 27,\ 36$ or $1,\ 2,\ 27,\ 36,\ 54$.*

Ideas for the proof

Let $G = A_6$ and $X = \mathrm{Cay}(G, S)$ be a connected tetravalent non-normal Cayley graph on G. Set $A = \mathrm{Aut}(X)$. By Fang et al. [13], we have the three cases:

Case 1: A is almost simple and $\mathrm{soc}(A)$ contains G as a proper subgroup and is transitive on $V(X)$.

Case 2: $G \rtimes \mathrm{Inn}(G) \leq A = (G \rtimes \mathrm{Aut}(G, S)) \cdot 2$ and S is a self-inverse union of G-conjugacy classes.

Case 3: A is not quasiprimitive on $V(X)$ and there is a maximal intransitive normal subgroup K of A such that one of the following holds:

(a) A/K is almost simple, $\mathrm{soc}(A/K)$ contains $GK/K \cong G$ and is transitive on $V(X_K)$;
(b) $A/K = \mathrm{AGL}_3(2)$, $G = \mathrm{PSL}(2, 7)$ and $X_K = K_8$;
(c) $\mathrm{soc}(A/K) \cong T \times T$ and $GK/K \cong G$ is a diagonal subgroup of $\mathrm{soc}(A/K)$.

Ideas for the proof

case 3 (a) holds. that is, A is not quasiprimitive on $V(X)$ and there is a maximal intransitive normal subgroup K of A such that A/K is almost simple, $\mathrm{soc}(A/K)$ contains $GK/K \cong G$ and is transitive on $V(X_K)$. Let B_i be the orbits of K on $V(X)$.

- X_K has valency 3 or 4.
- $\mathrm{PSU}(4,2)$ cannot be an subgroup of $\mathrm{Aut}(X_K)$.
- $\mathrm{soc}(A/K) = A_6$, $|B_i| = 2$ and X_K has valency 3.

Ideas for the proof

Now, we assume that $B_1 = \{1, h\}$ and $S = \{s_1, s_2, s_3, s_4\}$ where h is an involution. Then we have $s_1 \in C_G(h)$, $s_2 = s_1 h$ and $s_3^h = s_4$.

Since all involutions in A_6 are conjugate, one may take $h = (2\ 6)(3\ 5)$. By magma, we can get $S \cong \{(1\ 4)(2\ 3\ 6\ 5), (1\ 4)(2\ 5\ 6\ 3), (2\ 3\ 4\ 5\ 6), (2\ 6\ 5\ 4\ 3)\}$ or $\{(1\ 4)(2\ 3\ 6\ 5), (1\ 4)(2\ 5\ 6\ 3), (1\ 2)(5\ 6), (1\ 6)(2\ 3)\}$.

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Thanks!