Two fold orbital graphs and digraphs
Part 2

Josef Lauri (UoM), Russell Mizzi (UoM),
Raffaele Scapellato (Politecnico di Milano)

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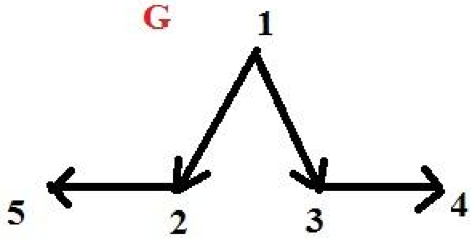
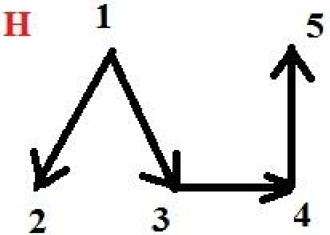
TF-isomorphic digraphs

$\alpha=(24)(1)(3)(5)$
$\beta=id$

(α,β) is a TF-isomorphism from G to H





Transposition of subgroups

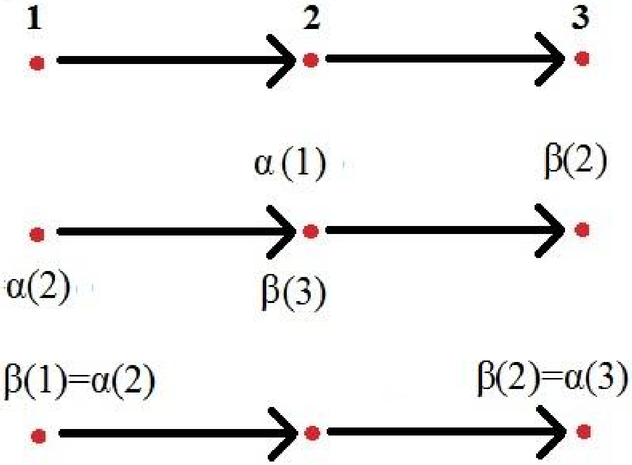
$\Gamma \leq S_n \times S_n$

$\Gamma^T = \{(\alpha,\beta) \in S_n \times S_n : (\beta,\alpha) \in \Gamma\}$

Γ is said to be *self-paired* if $\Gamma^T = \Gamma$

Is $\mathrm{Aut}^{TF}\, G$ self-paired?

Not necessarily: look at this example!



$\alpha=(12)(3)$ $\beta=(1)(23)$

(α,β) is a TF-automorphism

(β,α) is not!

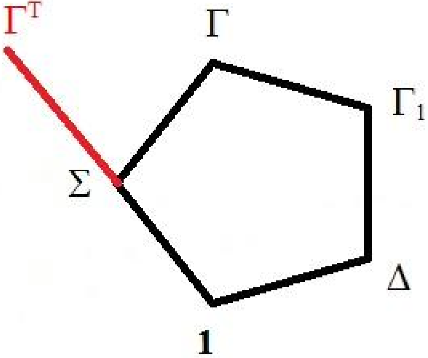
Subgroups of TF-automorphisms

$\Gamma = \mathrm{Aut}^{TF}\, G$

$\Gamma_1 = \{(\alpha,\beta) \in \mathrm{Aut}^{TF}\, G : \alpha,\beta \in \mathrm{Aut}\, G\}$

$\Delta = \{(\alpha,\alpha) \in S_n \times S_n : \alpha \in \mathrm{Aut}\, G\}$

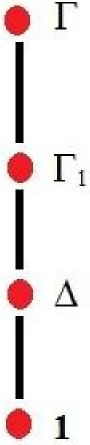
$\Sigma = \Gamma \cap \Gamma^T$



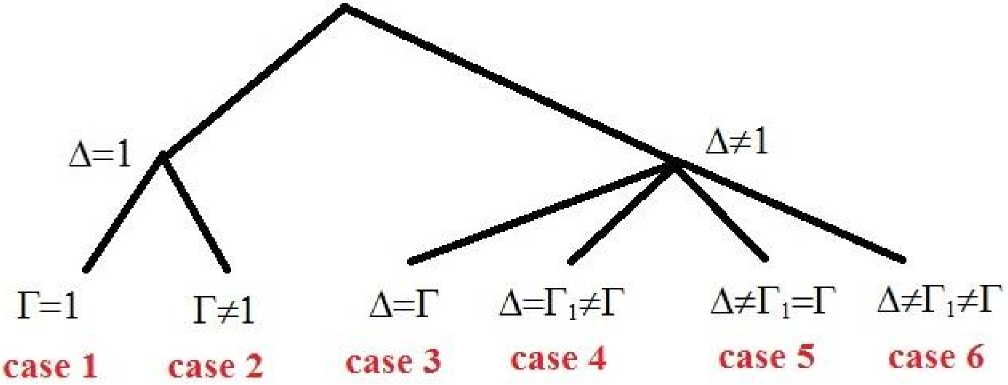
Subgroups of TF-automorphisms

If G is undirected, then $\Sigma = \Gamma$.

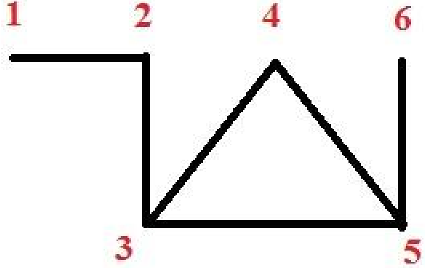
What further equalities can hold?



A case-by-case analysis of TF-automorphisms subgroups



A case-by-case analysis of TF-automorphisms subgroups

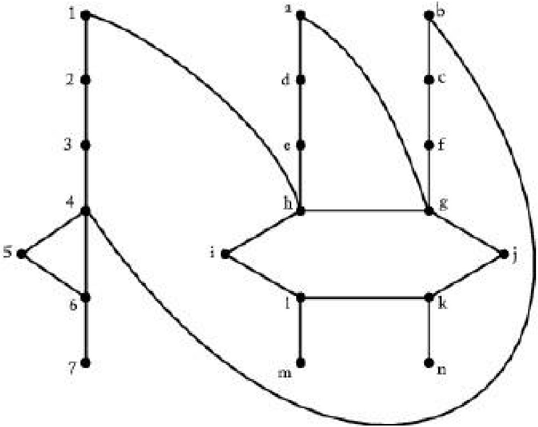


case 1

$\Delta=1 \quad \Gamma=1$

This graph is rigid and can easily be shown not to have TF-automorphisms either.

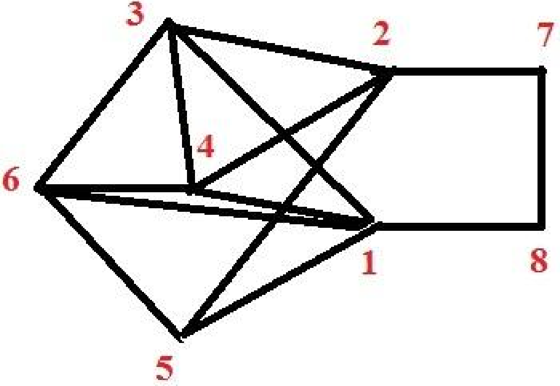
A case-by-case analysis of TF-automorphisms subgroups



case 2

$\Delta=1 \qquad \Gamma\neq1$

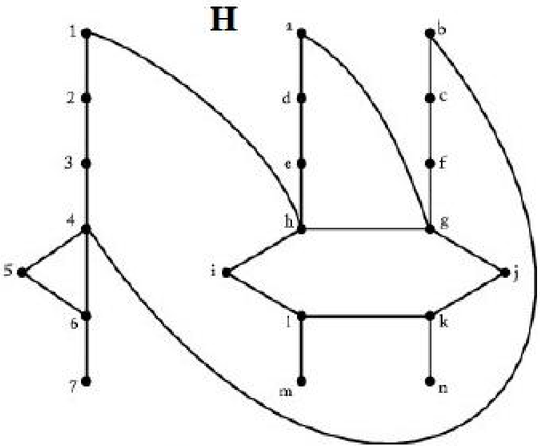
A case-by-case analysis of TF-automorphisms subgroups



case 3

$\Delta=\Gamma$

A case-by-case analysis of TF-automorphisms subgroups



case 4

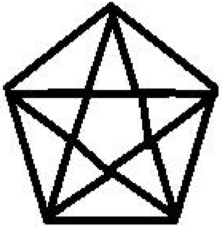
$\Delta = \Gamma_1 \neq \Gamma$

If G=B(H), then G satisfies this case.

A case-by-case analysis of TF-automorphisms subgroups

case 5

$\Delta \neq \Gamma_1 = \Gamma$



Every complete graph with at least two vertices lies in this case.

A case-by-case analysis of TF-automorphisms subgroups

case 6

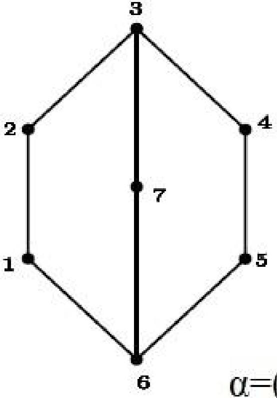
$\Delta \neq \Gamma_1 \neq \Gamma$

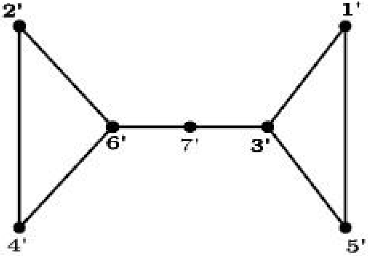
Take:
G' satisfying case 4: $\Delta = \Gamma_1 \neq \Gamma$
G" satisfying case 5: $\Delta \neq \Gamma_1 = \Gamma$
Let $G = G' \times G''$.
Then $\text{Aut } G' \times \text{Aut } G'' \leq \text{Aut } G$.

What properties are preserved by TF-isomorphisms?

Paths? No

Cycles? No





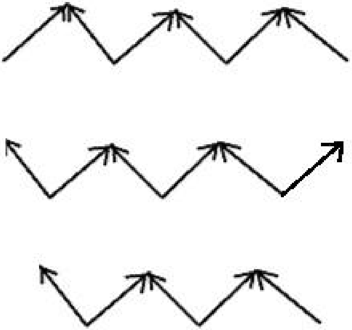
(α,β) is a TF-isomorphism

$\alpha=(25')(41')(36')(52')(14')(63')(77')$

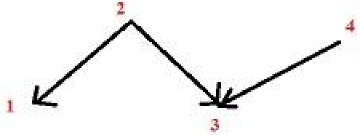
$\beta=(11')(22')(33')(44')(55')(66')(77')$

Alternating trails

Definition.
Let G be an oriented graph.
An *alternating trail* of G is a set of arcs of G that can be ordered as 10 12 32 34 54 56 ...
or as 01 21 23 34 45 65...
by a suitable labeling of vertices (the same vertex may appear more than once).

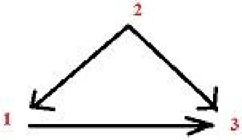


Alternating trails



Open

Starts at 1 ends at 4

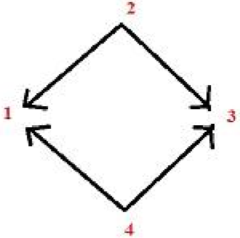


Half-open

Starts and ends at 1

Closed

Starts and ends at any vertex

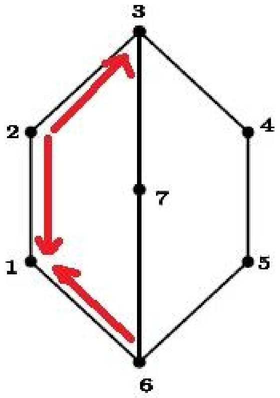


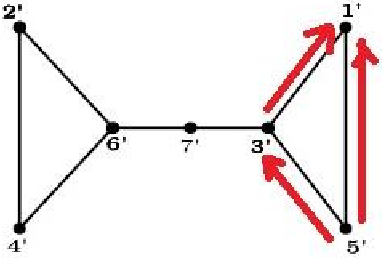
Alternating trails

Theorem. The pair (α,β) of bijections from $V(G)$ to $V(H)$ is a TF-isomorphism from G to H if and only if (α,β) preserves alternating trails.

TF-isomorphisms preserve also closed trails, but may swap open with half-closed trails.

Alternating trails





The first graph contains the open trail 61 21 23, corresponding to the semi-closed trail 31 51 53.

Conclusions

The conclusions of this talk on two-fold automorphisms are two-fold. There are in fact at least two directions worth investigating.

One is to study further the subgroups of $S_n \times S_n$ and their relationship with graphs, not limited to the vertex-transitive case.

Another is to give a closer look to alternating trails and their role, specifically in digraphs (not mixed).