Wilson's Graph Operations on Wada Dessins

Cristina Sarti

University of Frankfurt

August 6, 2010

Dessins d'enfants

Dessins d'enfants (=children drawings) are hypermaps on Riemann surfaces.

Hypermaps $\longrightarrow$ bipartite graphs drawn without crossings on a surface.

We say that a dessin on a surface X has signature $< p, q, r >$, if

$$\begin{aligned} p &:= \text{lcm of all valencies of the white vertices,} \\ q &:= \text{lcm of all valencies of the black vertices,} \\ 2r &:= \text{lcm of all valencies of the faces.} \end{aligned}$$

Dessin is uniform = all white vertices have the same valency p, all black vertices have the same valency q, all faces have the same valency $2r$.

Finite Projective Spaces

$$\begin{aligned} \mathbb{P}^m(\mathbb{F}_n) &= (\mathbb{F}_n^{m+1}\backslash\{0\})/\mathbb{F}_n^*, \qquad n = p^e, \quad p \text{ prime }, \\ &\cong \mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n. \end{aligned}$$

$$\ell = \frac{n^{m+1}-1}{n-1}, \qquad q = \frac{n^m - 1}{n-1}.$$

Each element of $\mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n$ can be written as a power g^i of a generating element $< g >$.

Notation:

$$g^b \leftrightarrow P_b \text{ (points)} \qquad g^w \leftrightarrow h_w \text{ (hyperplanes)} .$$

$\Longrightarrow$ we may number the points and the hyperplanes using integers $1, \ldots, \ell$.

Constructing the Dessins

Incidence pattern of points P_b and hyperplanes h_w using bipartite graphs.

Table: Conventions

point	black vertex $\bullet$
hyperplane	white vertex $\circ$
incidence	joining edge —

How do we know which point P_b is incident with which hyperplane h_w and vice versa? $\longrightarrow$ difference sets (Singer 1938).

Definition

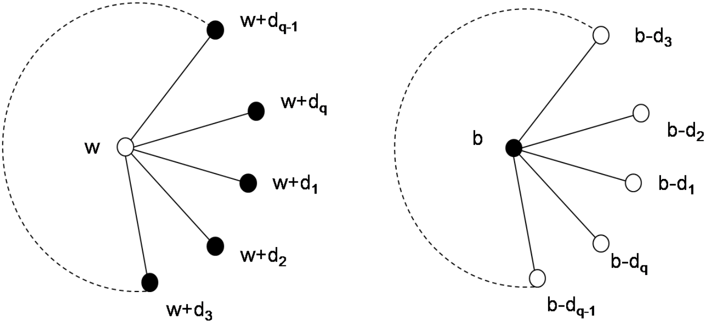
A (v, k, λ)-difference set $D = \{d_1, \ldots, d_k\}$ is a collection of k residues modulo v, such that for any residue $\alpha \not\equiv 0 \mod v$ the congruence

$$d_i - d_j \equiv \alpha \mod v$$

has exactly λ solution pairs (d_i, d_j) with d_i and d_j in D.

1. For projective spaces: $v = \ell$, $k = q$.
2. P_b and h_w are incident $\Leftrightarrow$ $b - w \equiv d_i \mod \ell$.
3. Any pair of points occur in λ different hyperplanes.

Local Incidence Pattern



Incidence relation: $b - w \equiv d_i \mod \ell$.

Wada Dessins

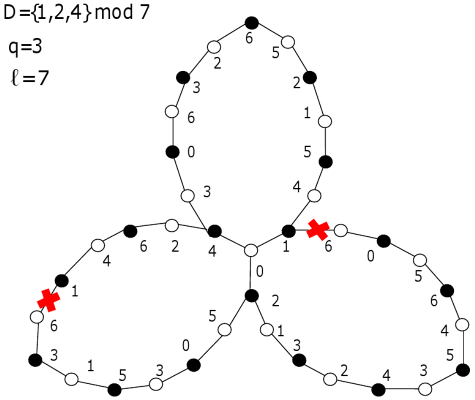
Construction of a uniform $< q, q, \ell >$-Wada dessin (Streit - Wolfart 2001) if the Wada condition is satisfied, i.e. if

$$(d_i - d_{i+1}, \ell) = 1 \; \forall i \in \mathbb{Z}/q\mathbb{Z} \; .$$

Wada property: each vertex of each color lies on the boundary of each cell.

$\longrightarrow$ Wada dessins are always uniform!

Example: $\mathbb{P}^2(\mathbb{F}_2)$ (Fano Plane)



Embedding of the bipartite graph in a Riemann surface (Klein's quartic in this case) $\Longrightarrow$ Wada dessin with signature $< 3, 3, 7 >$.

Automorphism Groups

Under special conditions the full automorphism group (orientation-preserving) of $< q, q, \ell >$-Wada dessins is the semidirect product $\Phi_f \ltimes \Sigma_\ell$ acting fixed point free on the edges.

Recall: $\mathbb{P}^m(\mathbb{F}_n) \cong \mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n$, $n = p^e$, i.e.

$$g^b \leftrightarrow P_b, \quad g^w \leftrightarrow h_w ,$$

$< g > :=$ generating element of $\mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n$.

1. Action of a Singer group $\Sigma_\ell \cong \mathbb{F}^*_{n^{m+1}}/\mathbb{F}^*_n$ (cyclic, of order ℓ):

$$\sigma^i : \quad \begin{array}{l} P_b \longmapsto P_{b+i} \ , \quad \sigma^i \in \Sigma_\ell \\ h_w \longmapsto h_{w+i} \ . \end{array}$$

On the Wada dessins the Singer automorphism acts permuting transitively the edges of type •—○ and of type ○—• on the cell boundaries.

2. Action of the group $\Phi_f \cong Gal(\mathbb{F}_{n^{m+1}}/\mathbb{F}_p)$, $f = e \cdot (m+1)$ generated by the Frobenius automorphism φ:

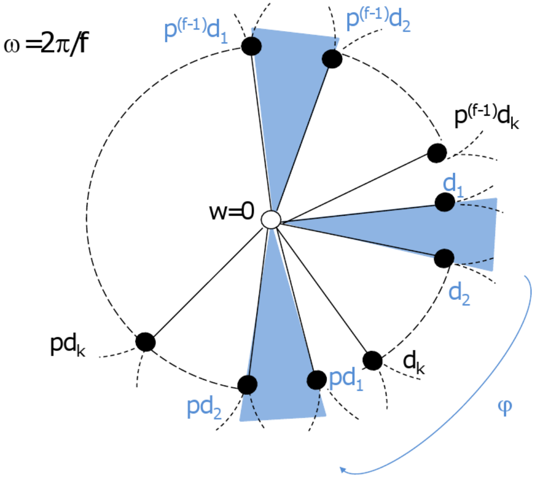
$$\varphi^k : \quad \begin{array}{l} P_b \longmapsto P_{b \cdot p^k} \ , \quad \varphi^k \in \Phi_f \\ h_w \longmapsto h_{w \cdot p^k} \ . \end{array}$$

The Frobenius automorphism acts by rotating the cells of the Wada dessins around the fixed vertices. The action on the cells is fixed point free.

BUT the Frobenius automorphism is **not** always an automorphism of Wada dessins $\longrightarrow$Frobenius Compatibility is needed!

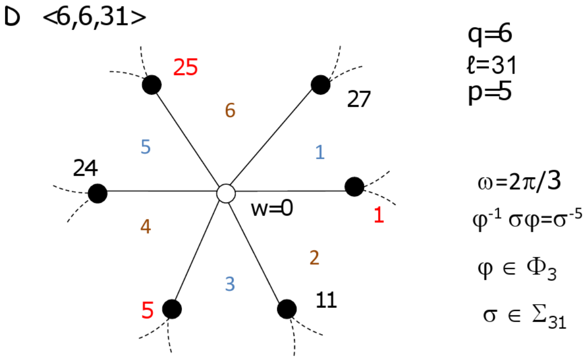
$$D_f = \{d_1, \cdots, d_k, \ pd_1, \cdots, pd_k, \ \cdots\cdots, p^{(f-1)}d_1, \cdots, p^{(f-1)}d_k\} \ ,$$
$$\frac{q}{f} = k \ .$$

Frobenius Compatibility

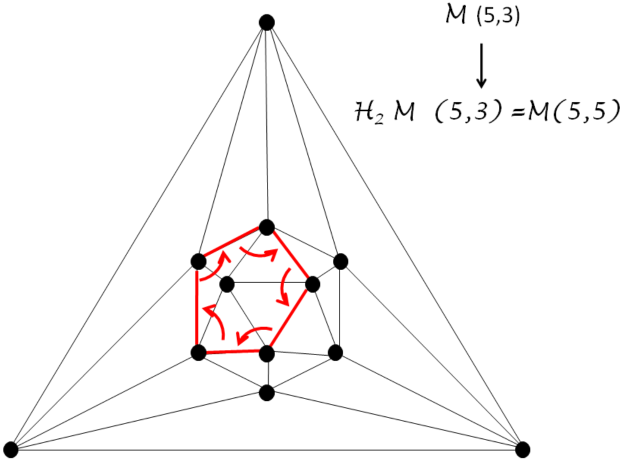


Example: $\mathbb{P}^2(\mathbb{F}_5)$

$$\mathbb{P}^2(\mathbb{F}_5)\ , \quad D_3 = \{1, 11, 5, 24, 25, 27\} \mod 31\ ,$$
$$\Sigma_{31}\ , \quad \Phi_3 \cong Gal(\mathbb{F}_{5^3}/\mathbb{F}_5)\ , \quad \Phi_3 \ltimes \Sigma_{31}\ .$$



Wilson Operations H_k



Wilson Operations and 'Mock' Wilson Operations

Under the action of a Wilson operator H_k on the underlying graph $\mathcal{G}$ of a map we obtain a new embedding of the graph on a Riemann surface. The cell valency may change.
The vertex valency is preserved ($\gcd(p, k) = 1$, p vertex valency).

$\longrightarrow$ 'mock' Wilson operators $H_{k,k}$ on $< q, q, \ell >$-Wada dessins:

$$H_{k,k} \longleftrightarrow \text{action of } H_k \text{ on the edges incident with the white and with the black vertices.}$$

The integer k is **not prime** to the vertex valency.

'Mock' Wilson Operations on Wada Dessins

We consider $< q, q, \ell >$-Wada dessins $\mathcal{D}$ with automorphism group:

$$\Phi_f \ltimes \Sigma_\ell \ ,$$

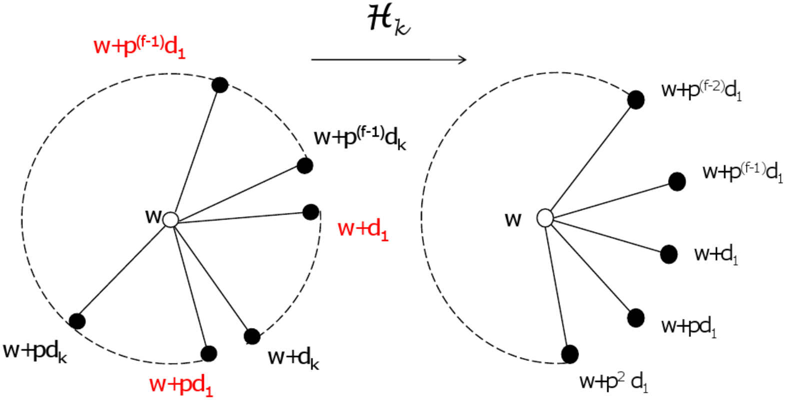
$\Phi_f \cong Gal(\mathbb{F}_{n^{m+1}}/\mathbb{F}_p)$, $n = p^e$,Σ_ℓ the Singer group.
Underlying difference set:

$$D_f = \{d_1, \cdots, d_k,\ pd_1, \cdots, pd_k,\ \cdots\cdots, p^{(f-1)}d_1, \cdots, p^{(f-1)}d_k\} \ ,$$
$$\frac{q}{f} = k \ . \quad (\text{Frobenius Compatibility})$$

$\longrightarrow$ 'mock' Wilson operators $H_{k,k}$ on them, with $k := \frac{q}{f}$.
'New' set:

$$H_k D_f = \{d_1, pd_1, \ldots, p^{(f-1)}d_1\} \ .$$

Local Incidence Pattern



Main Results

The resulting dessins are $< f, f, \ell >$-Wada dessins $H_{k,k}\mathcal{D}$ if the Wada property

$$((p^j d_1 - p^{j+1} d_1), \ell) = 1$$

is preserved.

1. The groups Σ_ℓ and Φ_f are still groups of automorphisms.
The group Σ_ℓ acts permuting transitively the edges of type $\bullet\!\!-\!\!\!-\!\!\circ$ and of type $\circ\!\!-\!\!\!-\!\!\bullet$ on the cell boundaries.
The group Φ_f acts rotating the cells around the fixed vertices, **BUT** now its action is not only free, it is also transitive.

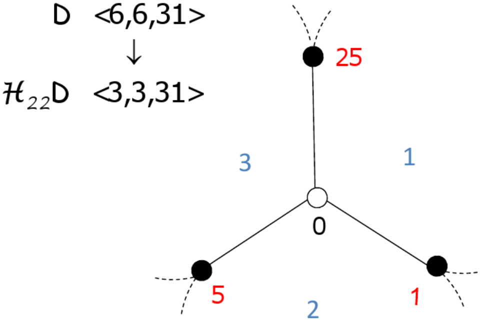
$\longrightarrow$ Consequence: the new dessins $H_{k,k}\mathcal{D}$ are regular with automorphism group $\Phi_f \ltimes \Sigma_\ell$.

2. $X :=$ surface of the embedding of the starting $< q, q, \ell >$-Wada dessin $\mathcal{D}$.
$Y :=$ surface of the embedding of the new $< f, f, \ell >$-Wada dessins $H_{k,k}\mathcal{D}$.
Due to group theoretical and function theoretical reasons we may prove that a ramified covering of Y by X exists. The covering is of degree k. Ramifications points are the black and the white vertices of $H_{k,k}\mathcal{D}$. The covering is unramified over the cell mid points.

Example: $\mathbb{P}^2(\mathbb{F}_5)$

$$\mathbb{P}^2(\mathbb{F}_5)\ , \qquad p = 5\ , \quad k = 2\ , \quad \Phi_3\ , \quad \Sigma_{31}.$$

$$D_3 = \{1, 11, 5, 24, 25, 27\} \mod 31\ ,$$
$$H_2 D_3 = \{1, 5, 25\} \mod 31\ .$$



Thank You!