A-LIKE ELEMENTS FOR A TRIDIAGONAL PAIR

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Overview

In this talk, we will begin by fixing a tridiagonal pair (A, A^*) on a vector space V, and we will define the notion of an A-like element of $End(V)$. In order to simplify the situation, we will introduce certain assumptions about the type of the tridiagonal pair (A, A^*).

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In this talk, we will begin by fixing a tridiagonal pair (A, A^*) on a vector space V, and we will define the notion of an A-like element of $End(V)$. In order to simplify the situation, we will introduce certain assumptions about the type of the tridiagonal pair (A, A^*).

A-like maps form a subspace of $End(V)$. We will provide a description of this space using certain decompositions of the underlying space V. Part of the task will be to decompose the space of A-like maps into smaller pieces and describe these pieces carefully. In the process we will obtain a basis for the space of A-like maps.

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Finally, we will consider a special basis for the underlying space V and state some results that relate this basis to the space of A-like maps.

Throughout this talk $\mathbb{F}$ denotes an algebraically closed field with characteristic 0, and V a nonzero finite-dimensional vector space over $\mathbb{F}$.

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(ii) There exists an ordering V_0, V_1, ..., V_d of the eigenspaces of A such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where

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(iii) There exists an ordering V_0^*, V_1^*, ..., V_d^* of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

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(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

From now on, we fix a tridiagonal pair (A, A^*).

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- This notion of standard orderings applies to A^* as well.

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  The sequence $(\rho_0, \rho_1, ..., \rho_d)$ is known as the shape of the tridiagonal pair.

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We would like to describe L. In order to simplify the situation, we will make some assumptions about our tridiagonal pair (A, A^*).

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From now on, we assume (A, A^*) has Krawtchouk type, with diameter $d = 2$, and shape $(1,2,1)$.

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In some sense, this is the smallest, simplest kind of TD pair that is not a Leonard Pair.

Two Decompositions of $E_1 V$

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It follows from work by Melvin Vidar (2008) that the subspace $E_1 V$ of V decomposes as follows:

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One can check that $E_1 V$ also has the following decomposition:

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Two Decompositions of V

Thus, the space V has the following two decompositions, each into 1-dimensional subspaces:

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Fact: In 2007, T. Ito and P. Terwilliger showed that there exists a nonzero bilinear form $\langle , \rangle$ on V such that $\langle Au, v\rangle = \langle u, Av\rangle$ and $\langle A^*u, v\rangle = \langle u, A^*v\rangle$ for all u, v in V. This form is unique up to multiplication by a scalar in the field. The form is nondegenerate and symmetric.

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It is not difficult to check that the above two decompositions of V are dual to one another with respect to this bilinear form.

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We use the above decompositions of V to describe L_0 more concretely:

A Description of L_0

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So $\{I, A, X^-, X^+\}$ forms a basis for L.

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Fact: It turns out that for all X in L and for all u, v in V, $\langle Xu, v \rangle = \langle u, Xv \rangle$.

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Therefore

$$L = L_0 \oplus Z(L)$$

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The Tetrahedron Algebra (Hartwig and Terwilliger)

Definition

Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \qquad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{ij}, x_{jk}] = 2x_{ij} + 2x_{jk}.$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[x_{ij}, [x_{ij}, [x_{ij}, x_{k\ell}]]] = 4[x_{ij}, x_{k\ell}].$$

The Tetrahedron Algebra and Krawtchouk type TD pairs

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Hartwig showed that there is a bijection between the set of isomorphism classes of Krawtchouk type TD pairs and the set of isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules. The 4-dimensional vector space V we are considering affords an action of $\boxtimes$, and the actions of A and A^* correspond to the actions of x_{12} and x_{03} respectively.

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Similarly, A^* acts as x_{03} on each of the 2-dimensional spaces, and it acts on V as $A^* \otimes I + I \otimes A^*$.

We call the above 2-dimensional modules W_1 and W_2.

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Then there exists a basis B_1 for W_1 and a scalar a, with respect to which the actions of A and A^* are

$$A : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A^* : \begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix}$$

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and there exists a basis B_2 for W_2 and a scalar b, with respect to which the actions of A and A^* are

$$A : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A^* : \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix}$$

If we tensor the bases B_1 and B_2 together, we can form a basis C for $V = W_1 \otimes W_2$ with respect to which the actions of A and A^* are as follows:

$$A : \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, A^* : \begin{pmatrix} 0 & a & b & 0 \\ 1/a & 0 & 0 & b \\ 1/b & 0 & 0 & a \\ 0 & 1/b & 1/a & 0 \end{pmatrix}$$

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Theorem

Each vector of the basis C is an eigenvector for the commutator $[A, A^*] = AA^* - A^*A$.

Other Elements of L

The maps $A \otimes I$ and $I \otimes A$ are elements of L. With respect to the basis C, they are represented by the following matrices

$$A \otimes I : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, I \otimes A : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Other Elements of L

Let the scalars s and c be defined as follows:

$$s = -\frac{1}{2}\left\{\frac{(a+1)(b+1)}{(a-b)}\right\}^2, \; c = \left(\frac{ab-1}{a-b}\right)^2$$

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Let $X_c = I + s\,(X^- + X^+)$. Then X_c is A-like, and with respect to the basis C, it is represented by the matrix

$$X_c : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1-c & c & 0 \\ 0 & c & 1-c & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Note that the set $\{I, A \otimes I, I \otimes A, X_c\}$ is a linearly independent set of elements of L, so this is a basis for L.

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Corollary

The map $\varphi_\eta : L \longrightarrow V$ is an isomorphism of vector spaces.