The schurity problem for quasi-thin association schemes

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3. for all $r, s, t \in S$ the intersection number

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Definition

A coherent configuration is called schurian if it the coherent configuration of some permutation group.

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2. the Tits theory on spherical buildings solves the schurity problem in a class of the Coxeter schemes (Z, 2005);
3. schemes of prime degree p: there exist non-schurian schemes of rank 3 but any scheme of rank $\geq (4p)^{4/5}$ is schurian (M-P, 2009).

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According to the Hanaki-Miamoto list there exist 1, 1 and 26 non-schurian quasi-thin schemes of degrees 16, 28 and 32 respectively.

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Theorem (H-M, 2002 and Z-M, 2008)

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Theorem (H-M, 2002 and Z-M, 2008)

A quasi-thin scheme is schurian whenever $|S^\perp| = 1$ and $S^\perp \subset S_1$, or $S^\perp \subset S_2$.

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2. any non-schurian quasi-thin scheme is a Klein scheme of index 4 or 7;
3. given $i \in \{4, 7\}$ there exist infinitely many non-schurian Klein schemes of index i.

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Theorem

Any non-Kleinian quasi-thin scheme is uniquely determined by its array of intersection numbers.