Finite vertex-quasiprimitive edge-transitive metacirculants

Jiangmin Pan

School of Mathematics and Statistics

Yunnan University, P. R. China

(Depending on joint works with C. H. Li)

Introduction

- Graphs here are simple, finite, connected, and undirected.

Introduction

- Graphs here are simple, finite, connected, and undirected.
- A graph Γ is called a metacirculant if there exists a metacyclic automorphism group $R \leq \mathsf{Aut}(\Gamma)$ which is transitive on the vertex set of Γ. By definition, Cayley graphs of metacycilic groups are metacirculants. But the inverse is not necessarily true.

Introduction

- Graphs here are simple, finite, connected, and undirected.
- A graph Γ is called a metacirculant if there exists a metacyclic automorphism group $R \leq \mathsf{Aut}(\Gamma)$ which is transitive on the vertex set of Γ. By definition, Cayley graphs of metacycilic groups are metacirculants. But the inverse is not necessarily true.
- The "metacirculants" was initiaed by Alspach and Parsons (1982), which provides a rich source of many interesting families of graphs. In particular, the following is a long-standing open problem in algebraic graph theory.

Introduction

- Graphs here are simple, finite, connected, and undirected.
- A graph Γ is called a metacirculant if there exists a metacyclic automorphism group $R \leq \mathsf{Aut}(\Gamma)$ which is transitive on the vertex set of Γ. By definition, Cayley graphs of metacycilic groups are metacirculants. But the inverse is not necessarily true.
- The "metacirculants" was initiaed by Alspach and Parsons (1982), which provides a rich source of many interesting families of graphs. In particular, the following is a long-standing open problem in algebraic graph theory.

  Problem A. Characterise edge-transitive metacirculants.

Some known results: Special classes of metacirculants

[1] B. Alspach, M. Conder, D. Marusic and M. Y. Xu, A classification of 2-arc-transitive circulants, *J. Algebraic Combin.* **5** (1996), no. 2, 83-86.

[2] I. Kovacs, Classifying arc-transitive circulants, *J. Algebraic Combin.* **20** (2004), 353-358.

[3] C. H. Li, Permutation groups with a cyclic regular subgroup and arc transitive circulants, *J. Algebraic Combin.* **21** (2005), 131-136.

Some known results (Continue)

[4] S. F. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, *J. Combin. Theory Ser. B* **98** (2008), 1349-1372.

[5] D. Marušič, On 2-arc-transitivity of Cayley graphs, *J. Combin. Theory Ser. B* **87** (2003), 162-196, and **96** (2006), 761-764.

[6] C. H. Li and H. Sims, On half-transitive metacirculant graphs of prime-power order, *J. Combin. Theory Ser. B* **81** (2001), 45-57.

[7] M. Y. Xu, Half-transitive graphs of prime-cube order, *J. Algebraic Combin.* **1** (1992), 275-282.

Some known results (Continue)

[8] D. Marušič and P. Šparl, On quartic half-arc-transitive metacirculants, *J. Algebraic Combin.* **28** (2008), 365-395.

[9] Chuixiang Zhou and Yan-Quan Feng, An infinite family of tetravalent half-arc-transitive graphs, *Discrete Math.* **306** (2006), 2205-2211.

[10] N. D. Tan, Cubic (m, n)-metacirculant graphs which are not Cayley graphs. *Discrete Math.* **154** (1996), no. 1-3, 237-244.

Classification

Theorem 1. Let Γ be a connected vertex-quasiprimitive edge-transitive metacirculants on V. Then one of the following holds:

(a) $\Gamma = \mathsf{K}_n$, $\mathsf{K}_n \times \mathsf{K}_n$, or $\mathsf{K}_n \Box \mathsf{K}_n$.

(b) $\Gamma = line(\mathsf{K}_p)$ and the complement.

(c) $\Gamma = \mathsf{Cay}(T, S)$, where $T = \mathrm{PSL}(2, p)$ and $S = \{g^T, (g^{-1})^T\}$ with $g \in T$, and G is of diagonal type.

(d) $G = \mathrm{PSL}(2, p)$, and Γ is described in Lemma 7-9.

(e) Four exceptional small groups:

$G = \mathrm{P\Gamma L}(2, 16)$, and $G_v = \mathrm{A}_5.\mathbb{Z}_4$, and Γ is arc-transitive and of valency 12, 15, or 40.

$G = \mathrm{P\Gamma L}(3, 4)$, $G_v = \mathrm{A\Gamma L}(1, 2^4)$, $|V| = 126$, and Γ is homomorphic to K_{21}.

$G = \mathrm{PSL}(5, 2)$, and Γ is the Grassmann graph $\mathrm{G}_2(5, 2)$ or its complement.

$G = \mathrm{PSU}(4, 2)$ or $\mathrm{PSU}(4, 2).2$, and Γ is the Schläfli graph or its complement.

(f) $\Gamma = \mathsf{Cay}(\mathbb{Z}_p^d, S)$ is a normal Cayley graph, where $p^d = p$, p^2, 3^4, 2^4, 2^6 or 2^8.

Observations

- Observation 1. Each edge-transitive metacirculant is a multi-cover of a vertex-quasiprimitive or a vertex biquasiprimitive edge-transitive metacirculant (basic graph).

Observations

- Observation 1. Each edge-transitive metacirculant is a multi-cover of a vertex-quasiprimitive or a vertex biquasiprimitive edge-transitive metacirculant (basic graph).
- Observation 2. Each basic edge-transitive metacirculant is a generalized orbital graph of a quasiprimitive or a biquasiprimitive permutation group containing a metacyclic transitive subgroup.

Observations

- Observation 1. Each edge-transitive metacirculant is a multi-cover of a vertex-quasiprimitive or a vertex biquasiprimitive edge-transitive metacirculant (basic graph).
- Observation 2. Each basic edge-transitive metacirculant is a generalized orbital graph of a quasiprimitive or a biquasiprimitive permutation group containing a metacyclic transitive subgroup.
- Thus Problem A (characterizes edge-transitive metacirculants) is tightly related with following Problem.

Observations

- Observation 1. Each edge-transitive metacirculant is a multi-cover of a vertex-quasiprimitive or a vertex biquasiprimitive edge-transitive metacirculant (basic graph).
- Observation 2. Each basic edge-transitive metacirculant is a generalized orbital graph of a quasiprimitive or a biquasiprimitive permutation group containing a metacyclic transitive subgroup.
- Thus Problem A (characterizes edge-transitive metacirculants) is tightly related with following Problem.

  Problem B. (Open problem of Wielandt 1949)
  Classifying quasiprimitive permutation groups containing a transitive metacyclic subgroup.

Research strategy

- Step 1. Classifying quasiprimitive permutation groups containing a transitive metacyclic subgroup.

Research strategy

- Step 1. Classifying quasiprimitive permutation groups containing a transitive metacyclic subgroup.
- Step 2. Classifying generalized orbital graphs of groups appeared in above step 1.

Solution of Wielandt's Problem

Theorem 2 [Li + Pan, 2009]. Let G be a finite quasiprimitive permutation group on Ω, and let R be a transitive metacyclic subgroup of G. Then one of the following holds:

(a) G is an almost simple group, and either $(G, G_\omega) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ or $(\mathrm{S}_n, \mathrm{S}_{n-1})$, or $(G, R, G_\omega) = (G, A, B)$ such that $R = A$ and $G_\omega = B$ as in Table I;

(b) $G = (\mathrm{PSL}(2,p) \times \mathrm{PSL}(2,p)).O$ where $O \le \mathbb{Z}_2^2$, R is regular, and either

$$p \equiv 3 \pmod 4,\ R \cong \mathbb{Z}_{\frac{p(p+1)}{2}}{:}\mathbb{Z}_{p-1} \cong (\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}) \times \mathrm{D}_{p+1}, \text{ or}$$
$$p \equiv 1 \pmod 4,\ O \ge \mathbb{Z}_2, \text{ and } R \cong \mathbb{Z}_{\frac{p(p+1)}{2}}{:}\mathbb{Z}_{p-1} \cong (\mathbb{Z}_p{:}\mathbb{Z}_{p-1}) \times \mathbb{Z}_{\frac{p+1}{2}};$$

(c) G is primitive of product action type of degree n^2 with socle T^2, and $R = \mathbb{Z}_n^2$ or $\mathbb{Z}_m^2{:}\mathbb{Z}_4$ with $m = \frac{n}{2}$ odd, and $T = \mathrm{A}_n$, or $\mathrm{PSL}(d,q)$ with $n = \frac{q^d-1}{q-1}$, or $(T, n) = (\mathrm{PSL}(2,11), 11)$, $(\mathrm{M}_{11}, 11)$ or $(\mathrm{M}_{23}, 23)$.

(d) G is affine, as described in Theorem HA.

TABLE I

row	G	A	B	conditions
1	S_p $\mathrm{A}_p.o$	$p{:}(p-1)$ $p{:}\frac{p-1}{2}$	$\mathrm{S}_{p-2},\ \mathrm{S}_{p-2}\times\mathrm{S}_2$ $\mathrm{S}_{p-2}\times o$	 $o\le 2$
2	$\mathrm{PSL}(d,q).o$ $\mathrm{PSL}(d,q).o.2$	$G(q^d).o_1$ $G(q^d).o_1.2$	$P_1.o_2$, parabolic $P_1.o_2$	where $q=p^f$, and $o_1o_2\cong o\le f.(d,q-1)$
3	$\mathrm{PGL}(2,p)$ $\mathrm{PSL}(2,p).o$	$p{:}(p-1)$ $p{:}\frac{p-1}{2}.o_1$	$\mathbb{Z}_{p+1},\ \mathrm{D}_{2(p+1)}$ $\mathrm{D}_{(p+1)o}$	 $o_1\le o\le 2,\ p\equiv 3\ (\mathrm{mod}\ 4)$
4	$\mathrm{PSL}(2,11).o$	$11{:}5.o_1$	$\mathrm{A}_4.o_2$	$o_1o_2=o\le 2$
5	$\mathrm{PSL}(2,11)$	$11,\ 11{:}5$	A_5	
6	$\mathrm{PSL}(2,29)$	$29{:}7$	A_5	
7	$\mathrm{PSL}(2,p)$ $\mathrm{PGL}(2,p)$	$p{:}\frac{p-1}{2}$ $p{:}(p-1)$	A_5 A_5	$p=11,19,29,59$
8	$\mathrm{PSL}(2,23)$ $\mathrm{PGL}(2,23)$	$23{:}11$ $23{:}22$	S_4 S_4	

(to be continued)

TABLE I (Continue)

9	$\mathrm{P\Gamma L}(2,16)$	17:8	$\mathrm{PSL}(2,4).4$	
10	$\mathrm{P\Gamma L}(3,4)$	$(7{:}3) \times \mathrm{S}_3$	$2^4{:}15{:}4$	
11	$\mathrm{PSL}(5,2).o$	$31{:}(5 \times o)$	$2^6{:}(\mathrm{S}_3 \times \mathrm{PSL}(3,2))$	$o \leq 2$
12	$\mathrm{PSU}(3,8).3^2.o$	$(3 \times 19{:}9).o_1$	$(2^{3+6}{:}63{:}3).o_2$	$o_1 o_2 = o \leq 2$
13	$\mathrm{PSU}(4,2).o$	$9{:}3.o_1$	$2^4{:}\mathrm{A}_5.o$	$o_1 \leq 2o \leq 4$
14	M_{11}	$11,\ 11{:}5$	$\mathrm{M}_{10},\ \mathrm{M}_9.2$	
15	M_{12} $\mathrm{M}_{12}.2$	6×2 D_{24}	M_{11} M_{11}	
16	$\mathrm{M}_{22}.2$	D_{22}	$\mathrm{PSL}(3,4).2$	
17	M_{23}	$23,\ 23{:}11$	$\mathrm{M}_{22},\ \mathrm{M}_{21}.2,\ 2^4.\mathrm{A}_7$	
18	M_{24}	D_{24}	M_{23}	

B-group

- A group R is called a *B-group* (named after Burnside) if each primitive permutation group containing R as a regular subgroup is necessarily 2-transitive.

B-group

- A group R is called a *B-group* (named after Burnside) if each primitive permutation group containing R as a regular subgroup is necessarily 2-transitive.
- Determining metacyclic B-groups has received much attention.

B-group

- A group R is called a *B-group* (named after Burnside) if each primitive permutation group containing R as a regular subgroup is necessarily 2-transitive.
- Determining metacyclic B-groups has received much attention.
- The next theorem give a classification of metacyclic B-group.

Classification of metacyclic B-groups

Theorem 3 [Li + Pan, 2009]. If a metacyclic group R is not a B-group, then R is isomorphic to one of the following groups, where p is a prime:

(a) $\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$, $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, $\mathbb{Z}_{29}{:}\mathbb{Z}_7$, $\mathbb{Z}_{57}{:}\mathbb{Z}_9$;

(b) $\mathbb{Z}_{\frac{p(p+1)}{2}}{:}\mathbb{Z}_{p-1} \cong (\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}) \times \mathrm{D}_{p+1}$, where $p \equiv 3 \pmod 4$;

$\mathbb{Z}_{\frac{p(p+1)}{2}}{:}\mathbb{Z}_{p-1} \cong \mathbb{Z}_{\frac{p+1}{2}} \times (\mathbb{Z}_p{:}\mathbb{Z}_{p-1})$, where $p \equiv 1 \pmod 4$;

(c) $\mathbb{Z}_n^2$, $\mathbb{Z}_m^2{:}\mathbb{Z}_4$ with m odd;

(d) $R = \mathbb{Z}_p$ or $\mathbb{Z}_p^2$, or $\mathbb{Z}_9{:}\mathbb{Z}_3$, $\mathbb{Z}_9{:}\mathbb{Z}_9$, or $\mathbb{Z}_8.\mathbb{Z}_2$, $\mathbb{Z}_4.\mathbb{Z}_4$, $\mathbb{Z}_8{:}\mathbb{Z}_8$ or $\mathbb{Z}_{16}{:}\mathbb{Z}_{16}$.

Coset graph

- Definition. Let G be a group, H a subgroup, and H a subset of G. The coset graph $\cos(G, H, HSH)$ is defined with the vertex set $[G : H]$, and Hx is adjacent to Hy iff $yx^{-1} \in HSH$.

Coset graph

- Definition. Let G be a group, H a subgroup, and H a subset of G. The coset graph $\cos(G, H, HSH)$ is defined with the vertex set $[G : H]$, and Hx is adjacent to Hy iff $yx^{-1} \in HSH$.
- Lemma 1. (1) $\Gamma = \cos(G, H, HSH)$ is connected iff $\langle H, S\rangle = G$;
(2) Γ is G-edge-transitive iff $\Gamma = \cos(G, G_\alpha, G_\alpha\{g, g^{-1}\}G_\alpha)$ for some $g \in G$, and $\mathsf{val}(\Gamma) = |G_\alpha : G_\alpha \cap G_\alpha^g|$ or $2|G_\alpha : G_\alpha \cap G_\alpha^g|$.

Coset graph

- Definition. Let G be a group, H a subgroup, and H a subset of G. The coset graph $\cos(G, H, HSH)$ is defined with the vertex set $[G : H]$, and Hx is adjacent to Hy iff $yx^{-1} \in HSH$.
- Lemma 1. (1) $\Gamma = \cos(G, H, HSH)$ is connected iff $\langle H, S\rangle = G$;
  (2) Γ is G-edge-transitive iff $\Gamma = \cos(G, G_\alpha, G_\alpha\{g, g^{-1}\}G_\alpha)$ for some $g \in G$, and $\mathsf{val}(\Gamma) = |G_\alpha : G_\alpha \cap G_\alpha^g|$ or $2|G_\alpha : G_\alpha \cap G_\alpha^g|$.
- Lemma 2. For a coset graph $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$, if all subgroups of H which are isomorphic to $H \cap H^g$ are conjugate in H, then there exists $f \in \mathbf{N}_G(H \cap H^g)$ such that $H\{g, g^{-1}\}H = H\{f, f^{-1}\}H$.

Proof of Theorem 1: Divided into four cases

- Let Γ be G-vertex-quasiprimitive edge-transitive metacirculants on V, and let $R \leq \mathsf{Aut}(\Gamma)$ be transitive on V. By Theorem 2, $G \leq \mathrm{Sym}(V)$ is of type diagonal, product action, almost simple or affine.

Proof of Theorem 1: Divided into four cases

- Let Γ be G-vertex-quasiprimitive edge-transitive metacirculants on V, and let $R \leq \mathsf{Aut}(\Gamma)$ be transitive on V. By Theorem 2, $G \leq \mathrm{Sym}(V)$ is of type diagonal, product action, almost simple or affine.
- We thus divide naturally our discussion into four cases.

Proof of Theorem 1: Divided into four cases

- Let $\mathit{\Gamma}$ be G-vertex-quasiprimitive edge-transitive metacirculants on V, and let $R \leq \mathsf{Aut}(\mathit{\Gamma})$ be transitive on V. By Theorem 2, $G \leq \mathrm{Sym}(V)$ is of type diagonal, product action, almost simple or affine.
- We thus divide naturally our discussion into four cases.
  - Case 1. G is of diagonal type (including simple diagonal or holomorph simple).

Proof of Theorem 1: Divided into four cases

- Let Γ be G-vertex-quasiprimitive edge-transitive metacirculants on V, and let $R \leq \mathsf{Aut}(\Gamma)$ be transitive on V. By Theorem 2, $G \leq \mathrm{Sym}(V)$ is of type diagonal, product action, almost simple or affine.
- We thus divide naturally our discussion into four cases.
  - Case 1. G is of diagonal type (including simple diagonal or holomorph simple).
  - Case 2. G is of product action type.

Proof of Theorem 1: Divided into four cases

- Let $\mathit{\Gamma}$ be G-vertex-quasiprimitive edge-transitive metacirculants on V, and let $R \leq \mathsf{Aut}(\mathit{\Gamma})$ be transitive on V. By Theorem 2, $G \leq \mathrm{Sym}(V)$ is of type diagonal, product action, almost simple or affine.
- We thus divide naturally our discussion into four cases.
  - Case 1. G is of diagonal type (including simple diagonal or holomorph simple).
  - Case 2. G is of product action type.
  - Case 3. G is almost simple.

Proof of Theorem 1: Divided into four cases

- Let Γ be G-vertex-quasiprimitive edge-transitive metacirculants on V, and let $R \leq \mathsf{Aut}(\Gamma)$ be transitive on V. By Theorem 2, $G \leq \mathrm{Sym}(V)$ is of type diagonal, product action, almost simple or affine.
- We thus divide naturally our discussion into four cases.
  - Case 1. G is of diagonal type (including simple diagonal or holomorph simple).
  - Case 2. G is of product action type.
  - Case 3. G is almost simple.
  - Case 4. G is affine.

Case 1. G is of diagonal type

Lemma 3. Let G be of diagonal type. Then $\mathsf{soc}(G) = T^2$ with $T = \mathrm{PSL}(2, p)$, and $R = \mathbb{Z}_{p^{\frac{p+1}{2}}}{:}\mathbb{Z}_{p-1}$ is regular on V, and the following statements hold:

(a) $\varGamma = \mathsf{Cay}(T, S)$, where S consists of a full conjugate class of elements g, g^{-1} of T;

(b) $\mathsf{Aut}\varGamma = (T \times T).2^2$, and $\varGamma$ is arc transitive;

(c) if $\varGamma$ is G-locally primitive, then $g = g^{-1}$ is an involution, and $\varGamma$ has valency $\frac{1}{2}p(p-1)$ or $\frac{1}{2}p(p+1)$.

Case 2. G is of product action type

Lemma 4. Let G be primitive of product action type. Then

(a) $\mathsf{soc}(G) = T^2$, where $T = \mathrm{A}_n$, $\mathrm{PSL}(d, q)$ with $n = \frac{q^d-1}{q-1}$, $\mathrm{PSL}(2, 11)$ with $n = 11$, M_{11} with $n = 11$, or M_{23} with $n = 23$.

(b) $\Gamma \cong \mathsf{K}_n \Box \mathsf{K}_n$ or $\mathsf{K}_n \times \mathsf{K}_n$, where $n \geq 5$.

(c) If Γ is G-locally primitive, then $\Gamma = \mathsf{K}_n \times \mathsf{K}_n$, and $T \neq \mathrm{PSL}(d, q)$ with $d \geq 3$.

Case 3. G is affine

Lemma 5. Let G be affine, with socle $\mathbb{Z}_p^d$. Then one of the following holds:

(a) G is 2-transitive, and $\Gamma = \mathsf{K}_{p^d}$;

(b) $d = 2e$, $G = H \wr \mathrm{S}_2$, where H is 2-transitive of degree p^e, and $\Gamma = \mathsf{K}_{p^e} \Box \mathsf{K}_{p^e}$ or $\mathsf{K}_{p^e} \times \mathsf{K}_{p^e}$;

(c) Γ is a normal Cayley graph of $\mathbb{Z}_p^d$.

Case 4. G is almost simple

We finally treat the almost simple case. Then (G, R) lies in TABLE 1 above. By determining generalized orbitals of all candidates there one by one, we finish the proof of Theorem 1 by following 6 lemmas.

Case 4. G is almost simple: (G is not 2-dimensional linear group)

Lemma 6. Suppose that $T = \mathsf{soc}(G)$ is not a 2-dimensional linear group. Then the following statements hold:

(1)$\Gamma = line(\mathsf{K}_p)$ or $\overline{line}(\mathsf{K}_p)$ with p prime, and one of the following holds:

(a) $T = \mathrm{A}_p$ or S_p, and $T_v = \mathrm{S}_{p-2}$ or $\mathrm{S}_{p-2} \times \mathrm{S}_2$, respectively;

(b) $p = 11$, $T = \mathrm{M}_{11}$, and $T_v = \mathrm{M}_9.2$;

(c) $p = 23$, $T = \mathrm{M}_{23}$, and $T_v = \mathrm{M}_{21}.2$ or $2^4{:}\mathrm{A}_7$.

(2) $G = \mathrm{P\Gamma L}(3, 4)$, Γ is homomorphic to K_{21}; moreover, Γ is not locally primitive.

(3) $G = \mathrm{PSL}(5, 2)$, and Γ is the Grassmann graph $\mathrm{G}_5(2)$ or the complement, of which only the former is locally primitive.

(4) $T = \mathrm{PSU}(4, 2)$ or $\mathrm{PSU}(4, 2).2$, and Γ is the Schläfli graph or its complement, of which only the former is locally primitive.

Case 4. G **is almost simple:** $\mathsf{soc}(G) = \mathrm{PSL}(2, p)$

Lemma 7. Let $\mathsf{soc}(G) = \mathrm{PSL}(2, p)$, $R = \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$, and $G_v \rhd \mathbb{Z}_{\frac{p+1}{2}}$. Then one of the following holds:

(a) $G = \mathrm{PGL}(2, p)$, $G_v = \mathbb{Z}_{p+1}$, Γ is of valency $p + 1$ or $2(p + 1)$.

(b) $G = \mathrm{PGL}(2, p)$, $G_v = \mathrm{D}_{2(p+1)}$, Γ is of valency $\frac{p+1}{2}$, unique; or Γ is of valency $p + 1$, one of $\frac{p-1}{2}$.

Moreover, if Γ is locally primitive, then $p \equiv 1 \pmod 4$, and Γ is of prime valency $\frac{p+1}{2}$.

Case 4. G is almost simple: $(G, G_v) = (\mathrm{PSL}(2, 11), A_4)$

Lemma 8. Let $T = \mathrm{PSL}(2, 11)$, and $T_v = \mathrm{A}_4$. Then

(a) Γ is $(G, 2)$-arc transitive and of valency 4, or

(b) G-arc transitive and of valency 6, or 12, or

(c) G-half-transitive of valency 12 or 24.

Moreover, if Γ is locally primitive, then Γ is 2-arc transitive of valency 4.

Case 4. G is almost simple: $(G, G_v) = (\mathrm{PSL}(2,p), A_5)$ with $p = 15, 29, 59$

Lemma 9. Let $G = \mathrm{PSL}(2,p)$, where $p = 19$, 29 or 59, and $G_v = \mathrm{A}_5$. Then one of the following holds:

(a) $G = \mathrm{PSL}(2,19)$, and either Γ is $(G,2)$-arc transitive of valency 6, or Γ is G-arc transitive of valency 20 or 30.

(b) $G = \mathrm{PSL}(2,29)$, and either Γ is G-arc transitive of valency 12 or 30, or Γ is G-half transitive of valency 40.

(c) $G = \mathrm{PSL}(2,59)$, and Γ has valency 6, 10, 12, 20, 24, 30 or 40.

Moreover, if Γ is locally primitive, then $G = \mathrm{PSL}(2,19)$ or $\mathrm{PSL}(2,59)$, and Γ is $(G,2)$-arc transitive of valency 6.

Case 4. G **is almost simple:** $(G, G_v) = (\mathrm{PSL}(2,23), S_4)$

Lemma 10. For $G = \mathrm{PSL}(2,23)$ and $G_v = \mathrm{S}_4$, the order $|V| = 253$, then

(a) $\varGamma$ is $(G,2)$-arc transitive of valency 4, or

(b) $\varGamma$ is G-arc transitive of valency 6 or 8, or

(c) $\varGamma$ is G-half transitive of valency 12 or 16.

If $\varGamma$ is G-locally primitive, then $\varGamma$ is $(G,2)$-arc transitive of valency 4.

Case 4. G is almost simple: $(G, G_v) = (\mathrm{P\Gamma L}(2, 16), (A_5 \times 2).2)$

Lemma 11. For $G = \mathrm{P\Gamma L}(2, 16)$ and $G_v = (\mathrm{A}_5 \times 2).2$, then $\varGamma$ is arc-transitive and of valency 12, 15 or 40, $\mathrm{Aut}\varGamma = G$, and $\varGamma$ is not locally primitive.

Thank You!