Map Enumeration Problem

Maps: Map is a 2-cell decomposition of a surface (preferably closed)

Category OrMaps: Maps on orientable surfaces together with orientation-preserving homomorphisms

Category Maps: Maps on surfaces (possibly with non-empty boundary) together with homomorphisms

Enumeration problem: Given property $\mathcal{P}$ determine a function $N_{\mathcal{P}}(e)$ counting the number of maps in $\mathcal{P}$ with e edges.

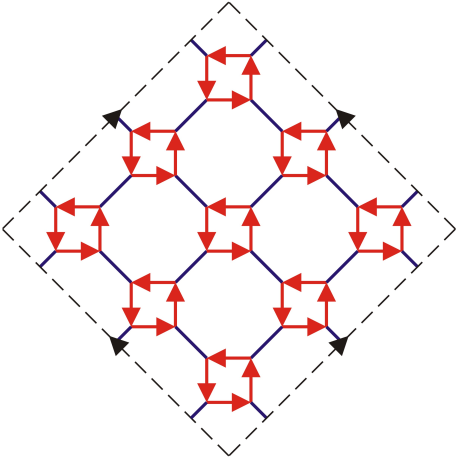
PROBLEMS CONSIDERED HERE:

$\mathcal{P}$ is the set of all ORMAPS,
$\mathcal{P}$ is the set of all MAPS,
$\mathcal{P}$ is the set of ORMAPS of given g,
$\mathcal{P}$ is the set of MAPS on a fixed S,

Why to do Map Enumeration?

1. **External motivation:** coming from chemistry, **statistical physics**, theory of strings, biology ...
2. **Outside combinatorics:** counting branched coverings of surfaces, counting subgroups of given index in given group, investigation of **action of absolute Galois group on maps**, algebraic curves ...
3. **Internal:** Map generation, asymptotic behavior of maps, learning more on maps, **Chiral versus Reflexible**,...

Feynmann diagrams



Brief (incomplete) History

1961-1970 **Tutte** founded the theory of map enumeration and derived generating functions for several families of rooted planar (spherical) maps,

1981 **Liskovets/Wormald** a formula for the number number of unrooted ormaps of genus 0,

1988-1992 **Bender, Canfield, Robinson, Arques ...** enumeration of rooted maps for other surfaces PP, torus, KB, ...

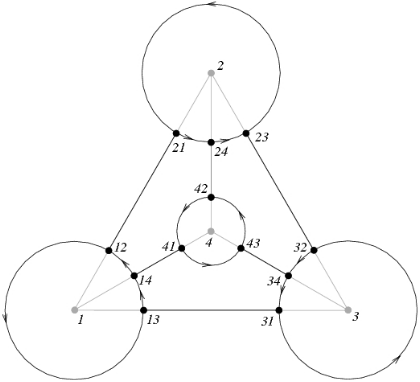
1995 - Further development of theory and new results by **D.M. Jackson, T.I. Visentin, M. Bousquet-Mélou, G. Schaeffer, T.R.S. Walsh, G. Jones, V. Liskovets ...** NOWADAYS MORE THAN 100 PAPERS PUBLISHED

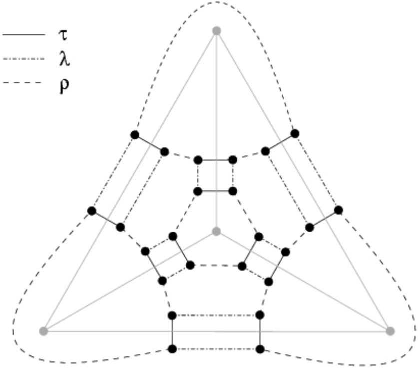
Combinatorial Maps

- ORMAPS: $(D; R, L)$, $R, L \in Sym(D)$, $L^2 = 1$, $\langle R, L \rangle$ is transitive on D,
- MAPS: $(F; r, \ell, t)$, $r, \ell, t \in Sym(F)$, $r^2 = \ell^2 = t^2 = (\ell t)^2 = 1$, $\langle r, \ell, t \rangle$ is transitive on F,
- ORDINARY MAPS: L, r, t, ℓ are fixed-point-free, but the category is not closed under taking quotients!

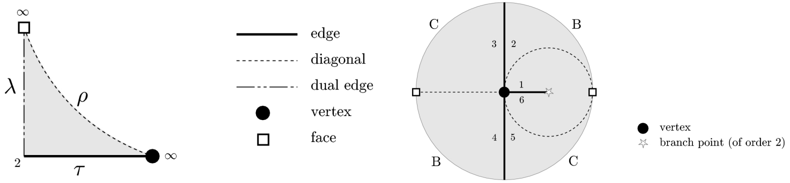
!!! Map = a particular action diagram of a permutation group = a particular 3-edge coloured cubic graph !!!

Combinatorial maps



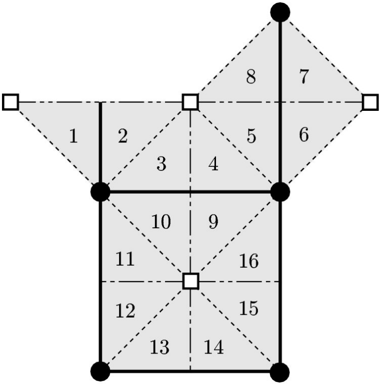


What is map? - back to geometry

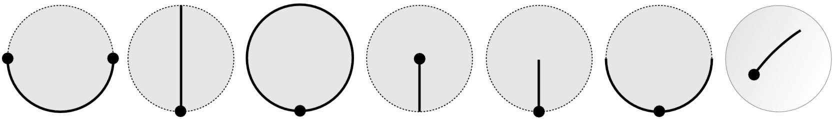


Theory of maps on Klein surfaces built by Bryant and Singerman

A map on disk



Seven maps based on two flags



Maps and subgroups of $\mathbb{Z} * \mathbb{Z}_2$, $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$

Orientable case(sensed maps)

$G = \mathbb{Z} * \mathbb{Z}_2 = \langle r, \ell | \ell^2 = 1 \rangle$

given map $(D; R, L)$ the mapping $r \mapsto R$, $\ell \mapsto L$
extends to a *transitive homomorphism* $G \to Sym(D)$.

G acts on D, a stabilizer H of this action is a subgroup of G of index $|D|$.

Vice-versa, given $H \leq G$ of finite index, we can set
$D = \{xH | x \in G\}$, and define $R(xH) = rxH$, $L(xH) = \ell xH$.

Maps and subgroups of $\mathbb{Z} * \mathbb{Z}_2$, $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$

OBSERVATIONS:

- the underlying surface is **compact** $=$ H is of **finite index**,
- map has **no semiedges** $=$ H is **torsion-free**,
- map is **connected** $=$ the respective homomorphism $G \to Sym(D)$ is **transitive**,
- the subgroups that correspond to ORMAPS are free.

Labelled and Rooted Maps

Rooted ormap An ormap with one dart distinguished as a root,

Rooted map A map with one flag distinguished as a root, on an orientable closed surface it is the same!!!

Labelled ormap An ormap with all darts distinguished,

Labelled map A map with all darts distinguished,

Dictionary:

Rooted ormap $=$ a torsion-free subgroup of $\mathbb{Z} * \mathbb{Z}_2$ of finite index,

Rooted map $=$ a torsion-free subgroup of $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$ of finite index,

Labelled ormap $=$ a transitive homomorphism $\mathbb{Z} * \mathbb{Z}_2 \to S_n$,

Labelled map $=$ a transitive homomorphism $\mathbb{Z} * (\mathbb{Z}_2 \times \mathbb{Z}_2) \to S_n$,

Isoclass of an ormap $=$ conjugacy class of a torsion-free subgroup of $\mathbb{Z} * \mathbb{Z}_2$,

Isoclass of a map $=$ conjugacy class of a torsion-free subgroup of $\mathbb{Z} * (\mathbb{Z}_2 \times \mathbb{Z}_2)$

WHY ROOTED MAPS? WHY LABELLED MAPS?
!!! **Action of $\mathrm{Aut}(M)$ is semiregular** !!!

Counting Group Homomorphisms

GENERAL SCHEME

Theorem

Let $G = \langle x_1, x_2, \ldots, x_r \rangle$ be a group. The number of homomorphisms B_n $G \to S_n$ satisfying $\mathcal{P}$ closed under conjugation and the number T_n of such transitive homomorphisms are related by

$$B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} T_{n-i} B_i,$$

$$\sum_{k=1}^{\infty} \frac{T_k}{k!} z^k = \log \sum_{k=1}^{\infty} \frac{B_k}{k!} z^k.$$

Stanley Vol. II, 1999, **Hurwitz** paper 1891, **M. Hall** in 1949.

Counting Labelled Sensed Maps Regardless of Genus

!!! **Labelled sensed map** (D, R, L) based on n darts $1, 2, \ldots, n$ is a **transitive homomorphism** $\mathbb{Z} * \mathbb{Z}_2 \to S_n$

Darts $D = \{1, 2, \ldots, n\}$

rotation R any permutation - $n! = (2e)!$ choices,

dart-reversing involution number of f. p. free involutions is $\frac{(2e)!}{2^e e!}$

Hence we have a direct formula for $B_{2e} = (2e)! \frac{(2e)!}{2^e e!}$ and we can apply the above recursive formula to derive T_{2e}.

!!! All actions is much easier to enumerate than transitive ones!!!

!!! Number of labelled maps $= (n-1)!$ number of rooted ones.

Counting Rooted Sensed Maps Regardless of Genus

Theorem

The number of rooted orientable maps $R^+(e)$ with e edges is given by the following equation

$$\sum_{e\geq 1} \frac{R^+(e)}{e} 2^{e-1} u^e = \log(\sum_{e\geq 0} \frac{(2e)!}{e!} u^e).$$

Proof. $T_{2e} = R^+(e)(2e-1)!$ and formula for T_{2e}.

(a) non-elementary proof by **D.M. Jackson and T.J. Visentin** in TAMS 332 (1990)

(b) another different proof by **D. Arques and J.S. Bacuteraud** DM 215 (2000)

Counting Rooted Maps Regardless of Genus

Labelled map on n flags is a transitive homomorphism $G \to S_n$, where

$G = \langle x, y, z | x^2 = y^2 = z^2 = (yz)^2 = 1 \rangle$.

There are no essentially new ideas but technically it is harder.

Theorem

The number of rooted maps (orientable or not) $R(e)$ with e edges is given by the following equation

$$\sum_{e \geq 1} \frac{R(e)}{e} 4^{2e-1} u^e = \log(\sum_{e \geq 0} \frac{(4e)!}{(2e)!\, e!} u^e).$$

Remark. The number $R^-(e)$ of rooted non-orientable maps with e edges is given by the formula $R^-(e) = R(e) - R^+(e)$.

Enumeration of rooted maps of given genus

Tutte 1963: the number of rooted planar maps with e edges is

$$\mathcal{N}_0(e) = \frac{2(2e)!3^e}{e!(e+2)!}$$

D. Arques 1987, Bender and Canfield 1988: Rooted toroidal maps

$$\mathcal{N}_1(e) = \sum_{k=0}^{e-2} 2^{e-3-k}(3^{e-1} - 3^k)\binom{e+k}{k}.$$

Exact formulas are known for $g = 2$ and $g = 3$ as well. Generally

the problem is not solved, although a lot of is known.

Problem: How to enumerate ISOCLASSES of Maps?

The only known result was done by **Liskovets, Wormald (1981)** for the sphere:

$$\Theta_0(e) = \frac{1}{2e}\left(\alpha(e) + \sum_{\substack{d|e\\ d<e}} \phi(e/d)\binom{d+2}{2}\mathcal{N}_0(d)\right)$$

$\alpha(e) = \mathcal{N}_0(e) + \binom{e}{2}\mathcal{N}_0(e/2–1)$ for e even,
$\alpha(e) = \mathcal{N}_0(e) + e\left(\frac{e-1}{2} + 2\right)\mathcal{N}_0((e-1)/2)$ for e odd.

N+Sasha Mednykh 2002 (first problem): **Can we derive a result for torus?**
Encyclopedia of integer sequences: first 6 numbers

Mednykh s lemma: From subgroups to conjugacy classes

RECALL the dictionary: ISOCLASS of (OR)MAPS = **conjugacy class of subgroups** in the universal group

Theorem (Mednykh)

Let Γ be a finitely generated group. Let $\mathcal{P}$ be a set of subgroups of Γ closed under conjugation. Then the number of conjugacy classes of subgroups of index n in $\mathcal{P}$ is given by the formula

$$N_{\Gamma}^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{K < \Gamma \\ [\Gamma : K] = m}} Epi_{\mathcal{P}}(K, Z_{\ell}).$$

$Epi_{\mathcal{P}}(K, Z_{\ell})$ - number of order preserving epimorphisms $\Gamma \to Z_{\ell}$ s. t. the kernel $\in \mathcal{P}$.

Example. Sensed maps on S_g with $n = 2e$ darts

The universal group is $\Gamma = \Delta(\infty, \infty, 2) = \langle x, y | y^2 = 1 \rangle$ and $\mathcal{P}$ is a set of subgroups of genus g and index n then

$$\sum_{\substack{K < \Gamma \\ [\Gamma:K]=m}} Epi_{\mathcal{P}}(K, Z_\ell) = \sum_{O \in Orb(S/Z_\ell)} h_O(m) Epi_0(\pi_1(O), Z_\ell),$$

where the second sum runs through all admissible cyclic orbifolds S_g/Z_ℓ. Hence the number of unrooted maps of genus g with e edges is

$$N_\Gamma^{\mathcal{P}}(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{O \in Orb(S/Z_\ell)} h_O(m) Epi_0(\pi_1(O), Z_\ell).$$

Problem A: $h_O(m) = ?$, **Problem B:** $Epi_0(\pi_1(O), Z_\ell) = ?$.

Axioms for ormaps orbifolds

(P1) if $x \in B$ then x is either an internal point of a face, or a vertex, or an end-point of a semiedge which is not a vertex,

(P2) each face contains at most one branch point,

(P3) the branch index of x lying at the free end of a semiedge is two.

Counting Rooted Ormaps (subgroups of given index) on closed orientable orbifolds

PROPOSITION M+N: Let $O = O[g; 2^{q_2}, \ldots, \ell^{q_\ell}]$ be an orbifold, $q_i \geq 0$ for $i = 2, \ldots, \ell$. Then the number of rooted maps $\nu_O(m)$ with m darts on the orbifold O is

$$\nu_O(m) = \sum_{s=0}^{q_2} \binom{m}{s} \binom{\frac{m-s}{2}+2-2g}{q_2-s, q_3, \ldots, q_\ell} \mathcal{N}_g((m-s)/2),$$

with a convention that $\mathcal{N}_g(n) = 0$ if n is not an integer, $\mathcal{N}_g(n)$ is the number of ordinary maps of genus g with n edges.

Fundamental group of cyclic 2-dim. orbifolds

For enumeration of sensed maps of given genus $\pi_1(O)$ is an F-group

$$\pi_1(M,\sigma) = F[\gamma; m_1, m_2, \ldots, m_r] =$$

$$\langle a_1, b_1, a_2, b_2, \ldots, a_\gamma, b_\gamma, e_1, \ldots, e_r |$$

$$\prod_{i=1}^{\gamma}[a_i, b_i]\prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \ldots e_r^{m_r} = 1\rangle.$$

N. of epimorphisms $\Pi_1(O) \to Z_\ell$

THEOREM $M + N$: Let $\Gamma = F[g; m_1, \ldots, m_r]$ be an $F-$group of signature $(g; m_1, \ldots, m_r)$ and $m = lcm\,(m_1, \ldots, m_r)$, $m|\ell$.
Then the number of order-preserving epimorphisms of the group Γ onto a cyclic group Z_ℓ is given by the formula

$$Epi_0(\Gamma, \mathrm{Z}_\ell) = m^{2g}\phi_{2g}(\ell/m)E(m_1, m_2, \ldots, m_r),$$

where

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m}\sum_{k=1}^{m}\Phi(k,\ m_1)\cdot\Phi(k,\ m_2)\ldots\Phi(k,\ m_r).$$

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ is a surface group of genus g we have

$$Epi_0(\Gamma, \mathrm{Z}_\ell) = \phi_{2g}(\ell).$$

Epi enumeration - main idea

$$|\mathrm{Hom}(G, \mathbb{Z}_\ell)| = \sum_{d|\,\ell} |\mathrm{Epi}(G, \mathbb{Z}_d)|,$$

Jordan function: $\varphi_p(\ell) = \sum_{d|\,\ell} \mu(\frac{\ell}{d})\, d^p$

VonSerneck function:

$$\Phi(x, n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})}\, \mu\left(\frac{n}{(x, n)}\right) = \sum_{\substack{1 \le k \le n \\ (k, n) = 1}} \exp(\frac{2\, ikx}{n}).$$

Reduction to homology group

For enumeration of sensed maps of given genus $\pi_1(O)$ is an F-group with relations:

$$\prod_{i=1}^{\gamma}[a_i, b_i]\prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \dots e_r^{m_r} = 1.$$

Reduction to epimorphisms from the homology group $H_1(O) \to \mathbb{Z}_\ell$

$$\{(a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_r) \in Z_d^{2g+r} :$$

$$x_1 + \dots + x_r = 0 \bmod d, (x_1, d) = d_1, \dots, (x_r, d) = d_r\},$$

where $d_i = d/m_i$. (restricted partition problem!!!)

Number of toroidal ormaps

The number of oriented unrooted toroidal maps with e edges is

$$\frac{1}{2e}\Big(\alpha(e) + \sum_{\ell|e} \phi_2(\ell)\mathcal{N}_1(e/\ell)\Big),$$

where

$\alpha(e) = \nu_{[0;2^4]}(e)$, if $e \equiv \pm 1, \pm 5 \mod 12$,
$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;2,4^2]}(e/2)$, if $e \equiv \pm 2, \pm 4 \mod 12$,
$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,3,6]}(e/3)$, if $e \equiv \pm 3 \mod 12$,
$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,4^2]}(e/2) + 2\nu_{[0;2,3,6]}(e/3)$, if $e \equiv 0, 6 \mod 12$.

Summary: Transfer from rooted to unrooted case

Mednykh + N. programme:

1. By Mednykh Lemma the problem of counting conjugacy classes decomposes into two separate problems of different nature:
2. Problem A of enumeration of rooted maps on admissible cyclic orbifolds, it is **purely combinatorial.**
3. Problem B of counting of the number of order preseving epimorphisms from fundamental groups of the above cyclic orbifolds onto cyclic groups, can be solved using techniques of **analytical number theory.**

Summary - Results

1. Exact formulae for the number of unrooted sensed maps of genus $g = 0, 1, 2, 3$ (Mednykh, Nedela 2006),
2. Our method gives an exact formula for the number of unrooted sensed maps of genus g, whenever rooted case is solved for genera $\gamma \leq g$
3. Number of sensed maps regardless of genus.
4. Number of maps regarless of genus.
5. Number of reflexible (chiral) maps regardless of genus.

Summary - Open problems

1. Derive exact formulae for the number of sensed maps of given genus.
2. Number of maps of given genus, not known even for the sphere. **WE CANNOT ENUMERATE ROOTED MAPS ON DISK.**
3. Asymptotic behavior of maps: Chiral versus reflexible.

Literature

1. J. H.Kwak, A. D. Mednykh and R. Nedela, Enumeration of orientable coverings of a nonorientable manifold, Discrete Math. Theor. Comput. Sci. Proc. AJ (2008), 215226 (electronic).

2. A. D. Mednykh, A new method for counting coverings over manifold with a finitely generated fundamental group, Dokl. Math. 74:1 (2006) 496502.

3.. Mednykh, Counting conjugacy classes of subgroups in a finitely generated group, J. Algebra 320:6 (2008), 22092217.

Literature

4. A. D. Mednykh and R. Nedela, Enumeration of unrooted maps with given genus, J. Combin. Theory Ser. B 96:5 (2006), 706729. MR2236507 (2007g:05088)
5. A. Mednykh and R. Nedela, Enumeration of unrooted hypermaps of a given genus, Discrete Math. 310:3 (2010), 518526.
6. Valery A. Liskovets, A multivariate arithmetic function of combinatorial and topological significance, Integers 10 (2010), 155-177.

n	Number of reflexible maps	Number of twins
01	2	0
02	5	0
03	20	0
04	85	11
05	418	226
06	2242	3597
07	12828	55006
08	77777	892791
09	493286	15763270
10	3260485	305360481
11	22314484	6483720916
12	157735801	150200835113
13	1147285362	3774756521566
14	8570960234	102339496556342
15	65611620808	2977913930684928

n	Number $U(n)$ of sensed maps with n edges
01	2
02	5
03	20
04	107
05	870
06	9436
07	122840
08	1863359
09	32019826
10	613981447
11	12989756316
12	300559406027
13	7550660328494
14	204687564072918
15	5955893472990664

edges, rooted maps on torus, unrooted maps on torus:

02, 1, 1
03, 20, 6
04, 307, 46
05, 4280, 452
06, 56914, 4852
07, 736568, 52972
08, 9370183, 587047
09, 117822512, 6550808
10, 1469283166, 73483256
11, 18210135416, 827801468
12, 224636864830, 9360123740
13, 2760899996816, 106189359544
14, 33833099832484, 1208328304864
15, 413610917006000, 13787042250528

TABLES

No. edges, No. rooted maps on torus, No. unrooted maps on torus:

19, 9083423595292949240, 239037464947999900
20, 110239596847544663002, 2755989928117365244
21, 1336700736225591436496, 31826208029615881656
22, 16195256987701502444284, 368074022535205870382
23, 196082659434035163992720, 4262666509741017440552
24, 2372588693872584957422422, 49428931123444048643388
25, 28692390789135657427179680, 573847815786545413529104