The normal quotient method for analysing the structure of highly symmetric graphs

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  - strongly regular graphs that are vertex- and edge-transitive.

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For any partition $\mathcal{P}$ of the vertices a graph Γ, the **quotient graph** *$\Gamma/\mathcal{P}$ is the graph with vertex set $\{P : P \in \mathcal{P}\}$ and vertices $P_1 \neq P_2$ adjacent iff there exist $v_1 \in P_1$ and $v_2 \in P_2$ with v_1 and v_2 adjacent in Γ.*

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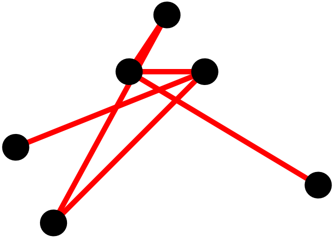

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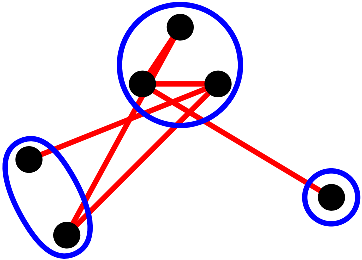

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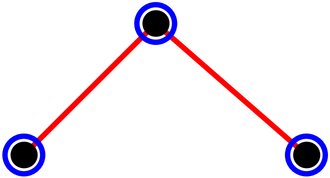

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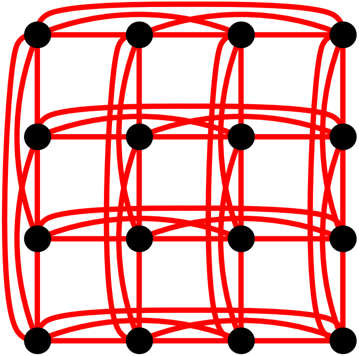

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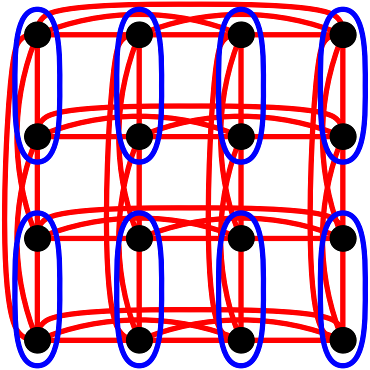

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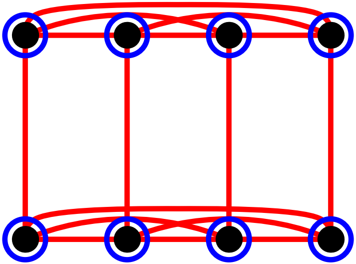

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If B is a G-block, the collection $\{g(B) : g \in G\}$ partitions the vertices of Γ, and each set in the partition is a G-block.

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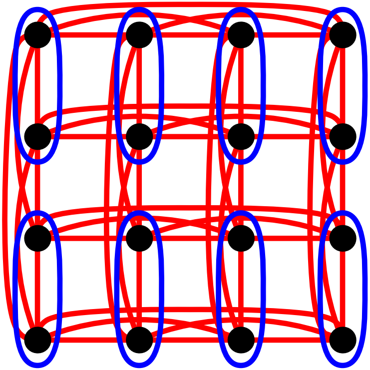

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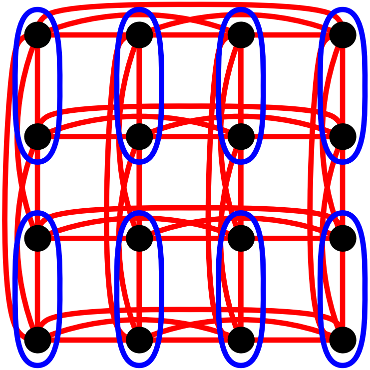

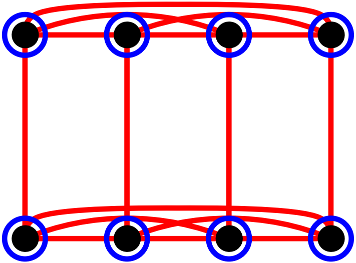

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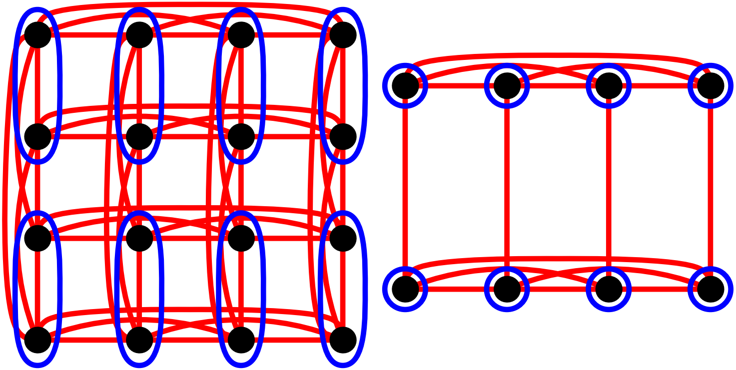

This makes it hard to prove that the quotient remains in the family.

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Then taking the orbits of N as the partition of the vertices of Γ produces a **normal quotient** graph, which has some nice properties.

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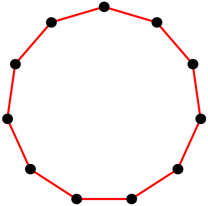

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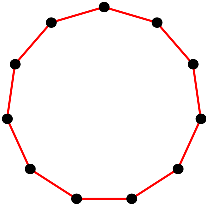

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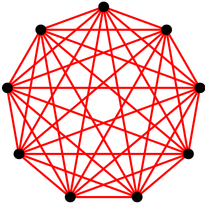

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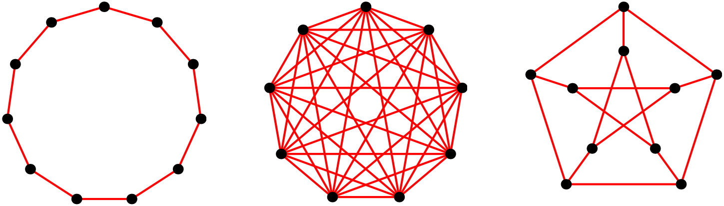

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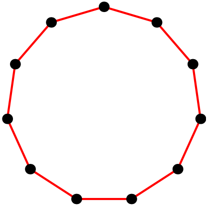

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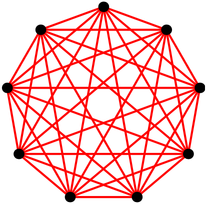

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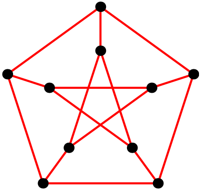

Odd graphs

also Johnson graphs $J(n, k)$.

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Complete classification is close (huge effort by many researchers).

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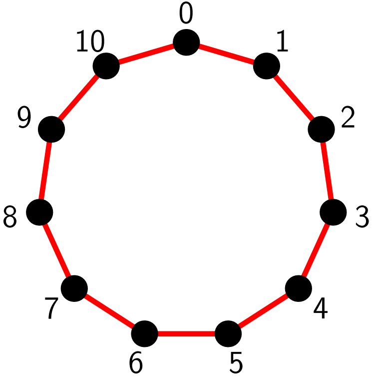

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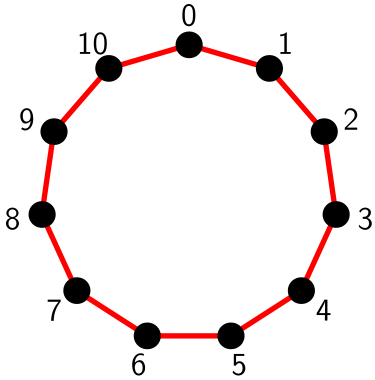

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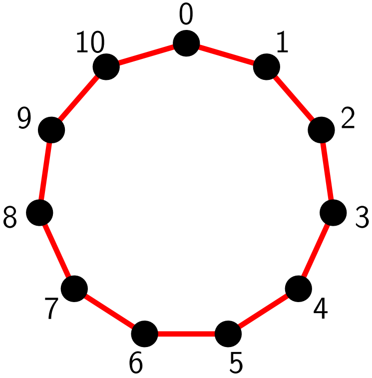

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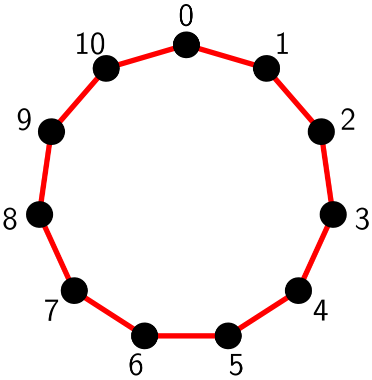

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but $(0, 1, 2, 3, 2, 1)$ is not a 5-arc.

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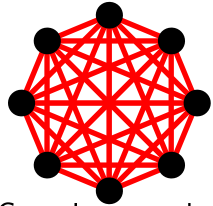

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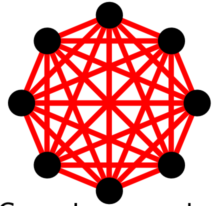

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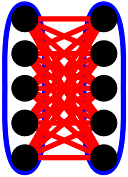

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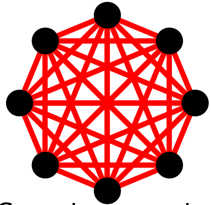

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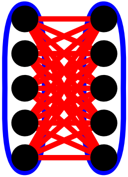
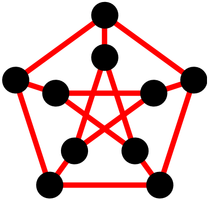


Complete bipartite graphs, $s = 3$



Odd graphs, $s = 3$ if $n \geq 3$

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Using normal quotients

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Irreducible graphs, continued

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Irreducible graphs, continued

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an s-arc-transitive graph will be irreducible (under the block quotient method) if and only if every group of its automorphisms that acts s-arc-transitively, is either quasiprimitive or bi-quasiprimitive.

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Definition

The graph Γ *with* $G \leq \mathrm{Aut}(\Gamma)$ *is* **locally** (G, s)**-arc-transitive** *if* G_v *acts s-arc-transitively on all s-arcs whose first vertex is* v*.*

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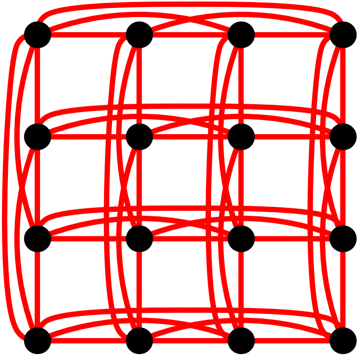
Example



A strongly regular graph with parameters $(16, 6, 2, 2)$.

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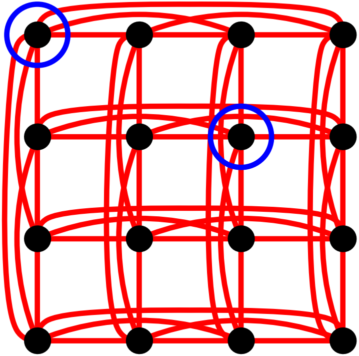

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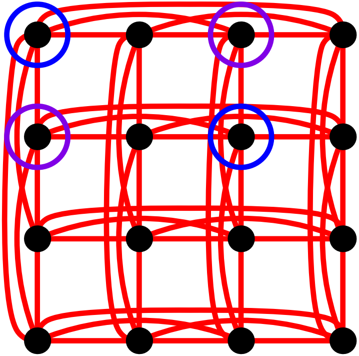

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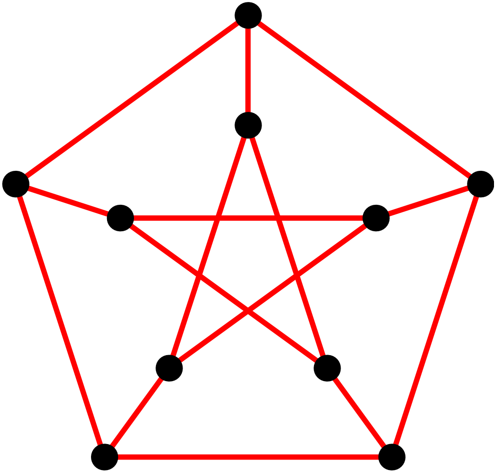

A strongly regular graph with parameters $(10, 3, 0, 1)$.

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  - Kneser graphs with $k = 2$; and
  - Paley graphs.
- There are also vertex-transitive, edge-transitive strongly regular graphs that are not rank 3; some Latin square graphs are examples.

Reduction works

Theorem (M., Praeger, Spiga, 2009)

A normal quotient of a vertex-transitive, edge-transitive strongly regular graph is vertex- and edge-transitive, and strongly regular.

Irreducible graphs

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We can reduce no further if either:

- we have a complete graph; or
- every vertex-transitive, edge-transitive subgroup G of $\mathrm{Aut}(\Gamma)$, is quasiprimitive.

Main result

Theorem (M., Praeger, Spiga, 2009)

If Γ is a strongly regular graph, and $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the vertices and edges, then G cannot be a holomorphic simple group.

Some other interesting results

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Theorem (M., Praeger, Spiga, 2009)

The graph $K_b \Box K_b$ has K_b as a normal quotient if and only if b is a prime power; otherwise, $K_b \Box K_b$ is irreducible.