Crosscovers

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Joint work with
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Symmetry in Graphs and Networks, II
Rogla, Slovenia
August, 2010

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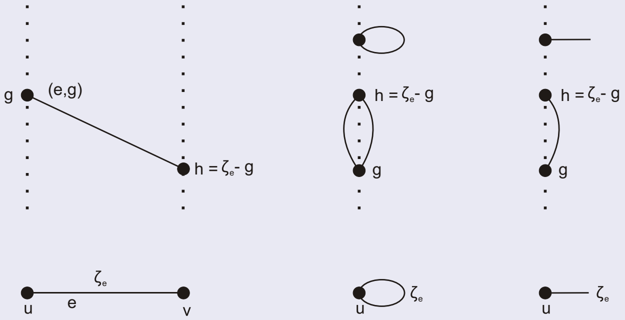
Derived crossover $\mathrm{Cr}(X, \zeta)$

- vertices $= V(X) \times \Gamma$, edges $= E(X) \times \Gamma$
- (e, g) connects (u, g) and $(v, \zeta_e - g)$



Example: Line graph of the Petersen graph

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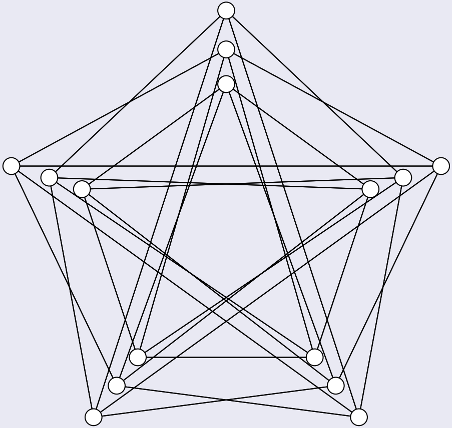
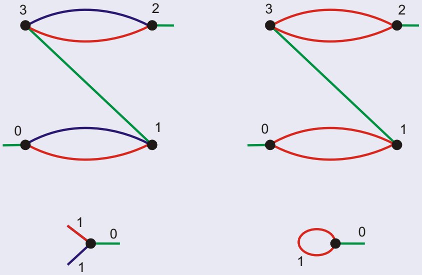
A $\mathbb{Z}_3$-crosscover of K_5



Example: Cayley crosscovers

Crosscovers of monopoles



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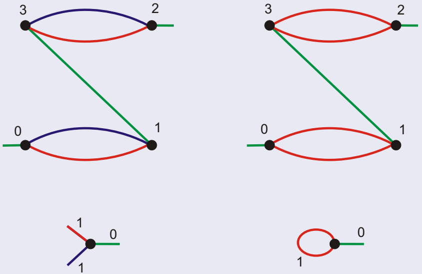

Cayley sum graphs

A subclass of Cayley crosscovers: $g, h \in \Gamma$ adjacent iff $g + h \in S$.

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- Certain irregular covers can be studied via abelian groups

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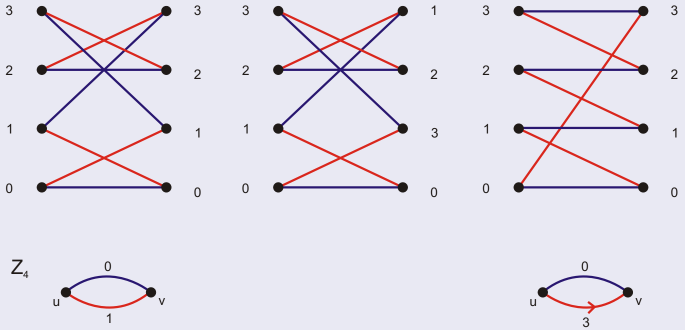

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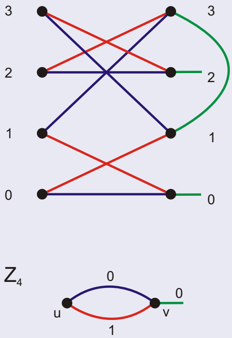

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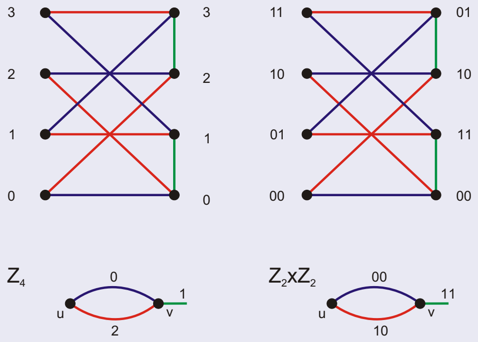

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