Edge Transitive Cayley Graphs of Square-free Order

Zai Ping Lu

Center for Combinatorics

Nankai University

Tianjin, 300071, China

email: lu@nankai.edu.cn

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$\Gamma = \mathsf{Cay}(G, S)$ is said to be normal if $G \lhd \mathsf{Aut}\Gamma$, and Γ is said to be normal edge transitive if $\mathbf{N}_{\mathsf{Aut}\Gamma}(G)$ is transitive on $E\Gamma$.

Several known results

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- $X = N{:}Y$ for some $Y \leq X$, where $N \lhd X$ is semiregular such that X/N acts faithfully on the N-orbits.
- Let T be a minimal normal subgroup of X. If T transitive, then T is simple, X has a regular subgroup $C{:}R$ for $R \leq G$ and $C = \mathbf{C}_X(T)$, and (T, T_α, R, C) is explicitly known.

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If $\mathcal{B}$ is the set of the N-orbits on $V\Gamma$ for some $N \lhd X$, then $\Gamma_{\mathcal{B}}$, denoted by Γ_N, is called a normal quotient of Γ, and Γ is called a normal cover of Γ_N if further Γ is a cover of Γ_N.

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Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph of square-free order. Then each quotient $\Gamma_{\mathcal{B}}$ is also a Cayley graph.

Theorem A

Let $\Gamma = \mathsf{Cay}(G, S)$ be an X-edge transitive Cayley graph of square-free order, $G \leq X \leq \mathsf{Aut}\Gamma$. Then

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(7) $\prod_i \mathbb{Z}_{p_i} \times \prod_j T_j \leq X \leq (\prod_i \mathrm{AGL}(1, p_i) \times \prod_j T_j).O$ for distinct primes p_i and non-isomorphic non-abelian simple groups T_j.

Locally primitive case

A graph Γ is said to be X-locally primitive if X_α induces a primitive permutation group on $\Gamma(\alpha)$ for each $\alpha \in V\Gamma$.

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(3) $X = N{:}Y$, Y is almost simple, $T := \mathsf{soc}(Y) \lhd X$, T has at most two orbits on $V\Gamma$, X has a subgroup $C{:}R$ acting regularly on each of T-orbits, where $C \le N$, $R \le G$, and all possible triples (T,R,C) are known.

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(1) Γ is isomorphic to one of well-defined graphs: the graphs in Cor 1 (1), and 8 graphs constructed from $\mathrm{AutPSL}(5,2)$, A_7, $\mathrm{PSL}(2,59)$, $\mathrm{PGL}(2,59)$ and M_{11}.

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(3) m-fold covers of the point-hyperplane (non-)incidence graph of $\mathrm{PG}(d-1,q)$, where $m > 1$ is a square-free divisor of $\frac{q-1}{(d,q-1)}$.

Theorem B

Let G be a group of square-free order and Γ be a connected edge transitive tetravalent Cayley graph of G. Then

(1) $\mathsf{Aut}\Gamma \cong G{:}\mathbb{Z}_2^2$ or $G{:}\mathbb{Z}_4$, Γ is arc regular and isomorphic a well-defined graph; or

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(4) Γ is t-transitive and $(\mathsf{Aut}\Gamma, G, \Gamma, t)$ is explicitly known.

Table 1.

Line	AutΓ	G	Γ	t
1	S_5	$\mathbb{Z}_5$	K_5	2
2	$\mathrm{S}_5 \times \mathbb{Z}_2$	$\mathbb{Z}_{10}, \mathrm{D}_{10}$	$\mathsf{K}_{5,5} - 5\mathsf{K}_2$	2
3	$\mathrm{S}_5 \times \mathrm{D}_6$	$\mathrm{D}_6 \times \mathbb{Z}_5, \mathrm{D}_{30}$	unique	2
4	$\mathrm{PGL}(2,7)$	$\mathbb{Z}_7{:}\mathbb{Z}_3$	$\mathrm{P}_{7,3}$	1
5	$\mathrm{PGL}(2,7) \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times (\mathbb{Z}_7{:}\mathbb{Z}_3), \mathbb{Z}_7{:}\mathbb{Z}_6$	$\mathrm{P}_{7,3}^{(2)}$	1
6	$\mathrm{PGL}(2,7) \times \mathrm{D}_{2l}$	$\mathbb{Z}_l \times (\mathbb{Z}_7{:}\mathbb{Z}_6), \mathrm{D}_{2l} \times (\mathbb{Z}_7{:}\mathbb{Z}_3)$	unique	1

3 $G = \langle c\rangle \times (\langle a\rangle{:}\langle b\rangle) \cong \mathbb{Z}_5 \times \mathrm{D}_6$, $S = \{cb, c^2ab, (cb)^{-1}, (c^2ab)^{-1}\}$.
$G = \langle a\rangle{:}\langle b\rangle \cong \mathrm{D}_{30}$, $S = \{ab, a^2b, a^4b, a^8b\}$.

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$G = \langle a, b \mid a^b = a^r, a^{7l} = b^6 = 1\rangle$, $S = \{b, ab, b^{-1}, (ab)^{-1}\}$.

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Line	$\mathsf{Aut}\Gamma$	G	Γ	s
1	$\mathrm{PSL}(3,2){:}\mathbb{Z}_2$	D_{14}	$\mathsf{K}_{7,7} - \mathrm{PG}(2,2)$	2
2	$\mathrm{PSL}(3,3){:}\mathbb{Z}_2$	D_{26}	$\mathrm{PG}(2,3)$	4
3	$\mathrm{PGL}(2,11)$	$\mathbb{Z}_{11}{:}\mathbb{Z}_5$	$\mathrm{P}_{11,5}$	2
4	$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	$\mathbb{Z}_2\times(\mathbb{Z}_{11}{:}\mathbb{Z}_5), \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$	$\mathrm{P}_{11,5}^{(2)}$	2
5	$(\mathrm{PSL}(2,11)\times\mathbb{Z}_3){:}\mathbb{Z}_2$	$\mathbb{Z}_3\times(\mathbb{Z}_{11}{:}\mathbb{Z}_5)$	unique	2
6	$\mathbb{Z}_2\times(\mathrm{PSL}(2,11)\times(\mathbb{Z}_3){:}\mathbb{Z}_2)$	$\mathbb{Z}_{33}{:}\mathbb{Z}_{10}, \mathbb{Z}_6\times(\mathbb{Z}_{11}{:}\mathbb{Z}_5)$	$\Gamma^{(2)}$	2
7	$\mathrm{PSL}(2,23)$	$\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$	$\mathrm{P}_{23,11}$	2
8	$\mathrm{PSL}(2,23)\times\mathbb{Z}_2$	$\mathbb{Z}_2\times(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})$	$\mathrm{P}_{23,11}^{(2)}$	2
9	$\mathrm{PSL}(2,23)\times\mathbb{Z}_2$	$\mathbb{Z}_2\times(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})$	two	2
10	$\mathrm{PSL}(2,23)\times\mathrm{D}_6$	$\mathrm{D}_6\times(\mathbb{Z}_{23}{:}\mathbb{Z}_{11})$	unique	2

line 1 $\mathsf{Cay}(G, \{b, ab, a^2b, a^4b\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathrm{D}_{14}$.

line 2 $\mathsf{Cay}(G, \{b, ab, a^3b, a^9b\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathrm{D}_{26}$.

line 3 $\mathsf{Cay}(G, \{a, a^{-1}, b^2, b^3\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5$ with $a^b = a^3$.

line 4 $\mathsf{Cay}(G, \{ab^5, a^{-1}b^5, b, b^{-1}\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_{10}$ with $a^b = a^2$.
$\mathsf{Cay}(G, \{ac, a^{-1}c, b^2c, b^3c\})$ for $G = \langle a \rangle{:}\langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5 \times \mathbb{Z}_2$ with $a^b = a^3$.

line 5 $\mathsf{Cay}(G, \{ac, a^{-1}c^{-1}, b^2, b^3\})$ for $G = \langle a \rangle{:}\langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5 \times \mathbb{Z}_3$ with $a^b = a^3$.

line 6 $\mathsf{Cay}(G, \{ac, a^{10}c^5, b^2c^3, b^3c^3\})$ for $G = \langle a \rangle{:}\langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{11}{:}\mathbb{Z}_5 \times \mathbb{Z}_6$ with $a^b = a^3$;
$\mathsf{Cay}(G, \{ab^5, a^{-1}b^5, b, b^9\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathbb{Z}_{33}{:}\mathbb{Z}_{10}$ with $a^b = a^2$.

line 7 $\mathsf{Cay}(G, \{ab, (ab)^{-1}, b^4, b^7\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{11}$ with $a^b = a^2$.

line 8 $\mathsf{Cay}(G, \{abc, (ab)^{-1}c, b^4c, b^7c\})$ for $G = \langle a \rangle{:}\langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{11} \times \mathbb{Z}_2$ with $a^b = a^2$.

line 9 $\mathsf{Cay}(G, \{abc, (ab)^{-1}c, b^4, b^7\})$ and $\mathsf{Cay}(G, \{ab, (ab)^{-1}, b^4c, b^7c\})$ for $G = \langle a \rangle{:}\langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{11} \times \mathbb{Z}_2$ with $a^b = a^2$.

line 10 $\mathsf{Cay}(G, \{ab, (ab)^{-1}, a^{23}b^4, (a^{23}b^4)^{-1}\})$ for $G = \langle a \rangle{:}\langle b \rangle \cong \mathbb{Z}_{69}{:}\mathbb{Z}_{22}$ with $a^b = a^2$.

Thank You!