Finite Edge-transitive Graphs

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Lemma.

If $\Gamma = (V, E)$ is G-edge-transitive, then one of the following holds:

(a) *Γ is G-arc-transitive;*

(b) *G is trans on both V and E, but intrans on the arc set;*

(c) *G is intransitive on V; so Γ is bipartite.*

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- **(c)** *G is intransitive on V; so Γ is bipartite.*

Remarks:

- Graphs in item (b) are called G-half-arc-transitive;
- Regular graphs in item (c) are G-semisymmetric.

Quotient

Let $\Gamma = (V, E)$ be G-edge-transitive.

We consider block quotients and normal quotients.

Let $\mathcal{B}$ be a non-trivial G-invariant partition of V. Then the corresponding block quotient is denoted by $\Gamma_{\mathcal{B}}$. If $\mathcal{B}$ is the set of a normal subgroup $N \lhd G$, then denote $\Gamma_{\mathcal{B}}$ by Γ_N.

Observation

- *If Γ is edge-transitive, then so is $\Gamma_{\mathcal{B}}$.*
- *If Γ is arc-transitive, then so is $\Gamma_{\mathcal{B}}$.*

Covers

Definition

The graph Γ is a cover of $\Gamma_{\mathcal{B}}$ if, for two adjacent blocks $B, C \in \mathcal{B}$, the induced subgraph $[B, C]$ is a perfect matching.

'Cover' would be the most important relation between Γ and $\Gamma_{\mathcal{B}}$.

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Naturally, we would ask

Question

If Γ and $\Gamma_{\mathcal{B}}$ have the same valency, is Γ a cover of $\Gamma_{\mathcal{B}}$?

Covers

The answer is positive for some special cases

- If Γ and Γ_N have the same valency, then Γ is a cover of Γ_N.
- If Γ is *G*-locally-primitive, and Γ and $\Gamma_{\mathcal{B}}$ have the same valency, then Γ is a cover of $\Gamma_{\mathcal{B}}$.

A graph Γ is *G*-locally-primitive if $G \leq \mathrm{Aut}\Gamma$ is such that G_v acts primitively on $\Gamma(v)$ for each vertex v.

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However, generally,

The answer is negative!

Example

Construction (Li-Praeger-Zhou)

Let $G = \mathrm{PSL}(2,p)$, where $p \equiv 1 \pmod{16}$. Let H be a Sylow 2-subgroup of G.

Then $H = \langle a\rangle{:}\langle b\rangle \cong \mathrm{D}_{16}$, $\langle a^4, b\rangle \cong \mathbb{Z}_2^2$, and $\mathbf{N}_G(\langle a^4, b\rangle) = \mathrm{S}_4$.

Some involution $g \in \mathbf{N}_G(\langle a^4, b\rangle)$ interchanges a^4 and b.

Let $L = \langle a^4, ba\rangle \cong \mathbb{Z}_2^2$, and define

$$\Sigma = \mathrm{Cos}(G, H, HgH), \quad \Gamma = \mathrm{Cos}(G, L, LgL).$$

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Theorem.

- *Both* Σ *and* Γ *are* G*-symmetric of valency* 4*,*
- Σ *is a block quotient of* Γ*, and*
- Γ *is not a cover of* Σ*.*

Normal covers

If Γ is a cover of Γ_N, then Γ is called a normal cover of Γ_N, or a regular cover.

Observation

If $\Gamma = (V, E)$ is a cover of Γ_N, then N is semiregular on V and G/N is faithful on $V\Gamma_N$.

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The answer is negative!

Example*

Let $R = \langle a_1, \ldots, a_6 \rangle \cong \mathbb{Z}_4^6$, and let $S = \{a_i, a_i^{-1} \mid 1 \le i \le 6\}$.

Then $\mathrm{Aut}(R)$ has a subgroup $H \cong \mathrm{A}_5$ which acts transitively on S with a block system $\{\{a_i, a_i^{-1}\} \mid 1 \le i \le 6\}$ and the stabiliser of the block $\{a_6, a_6^{-1}\}$ equals $\langle x, y \rangle \cong \mathrm{D}_{10}$, where

$$x = (a_1 a_2 a_3 a_4 a_5)(a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1}), \text{ and}$$
$$y = (a_1 a_1^{-1})(a_2 a_5^{-1})(a_3 a_4^{-1})(a_2^{-1} a_5)(a_3^{-1} a_4)(a_6 a_6^{-1}).$$

Let $G = \hat{R}{:}H$, and $\Gamma = \mathrm{Cay}(R, S)$. Then Γ is symmetric of valency 12 as $H \le \mathrm{Aut}(R, S)$ is trans on S.

The group G has a normal subgroup $N = \langle \hat{a}_i^2 \mid 1 \leq i \leq 6 \rangle \cong \mathbb{Z}_2^6$.

The quotient graph Γ_N is a Cayley graph of $\mathbb{Z}_2^6$ of valency 6.

Furthermore, N is the kernel of G acting on $V\Gamma_N$ and is semiregular on the vertex set R.

However, Γ is not a cover of Γ_N.

Cayley graphs

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A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called normal edge-transitive if $\mathbf{N}_{\mathrm{Aut}\Gamma}(\hat{G})$ is transitive on the edge set of Γ. Since

$$\mathbf{N}_{\mathrm{Aut}\Gamma}(\hat{G}) = \hat{G}{:}\mathrm{Aut}(G, S),$$

$\mathrm{Aut}(G, S)$ is transitive on $\{\{s, s^{-1}\} \mid s \in S\}$.

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Theorem.

Every (connected) vertex- and edge-transitive graph has a normal cover which is a (connected) normal edge-transitive Cayley graph.

Classes of graphs closed under normal quotients

Classes of graphs closed under normal quotients

$\{$ s-arc-transitive graphs$\}$, with $s \geq 1$; (Praeger 1990's)

$\{$locally s-arc-transitive graphs$\}$, with $s \geq 1$;
$\{$locally-primitive graphs$\}$; (Giudici-Li-Praeger, 2002-2006)

$\{$vertex-transitive locally-quasiprimitive graphs$\}$;
(Li-Praeger-Venkatesh-Zhou, 2002)

$\{$2-path-transitive graphs$\}$; (Hua Zhang's PhD project)

A graph Γ is G-locally-quasiprimitive if $G \leq \mathrm{Aut}\Gamma$ is such that G_v acts quasiprimitively on $\Gamma(v)$ for each vertex v, ie, each normal subgroup of G_v is trivial or trans on $\Gamma(v)$.

More graphs

$\{$edge-transitive graphs of odd order$\}$; (Jingjian Li's PhD project)

$\{$edge-transitive graphs of square-free order$\}$; (Gaixia Wang' PhD project)

$\{$metacirculants$\}$, (Jiangmin Pan's PhD project);

$\{$underlying graphs of regular maps$\}$, (Li-Siran).

$\{$vertex-, edge-trans strongly regular graphs$\}$, (Morris-Praeger-Spiga).

$\{$locally distance-transitive graphs$\}$, (see next talk by Giudici).

O'Nan-Scott type theorem

Theorem (Praeger 1993)

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O'Nan-Scott type theorem

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Let $G \leq \mathrm{Sym}(V)$ be quasiprimitive. Then G has ≤ 2 minimal normal subgroups, and one of the following holds:

- **HA** $\mathrm{soc}(G)$ *is an abelian minimal normal subgroup, and* $G \leq \mathrm{AGL}(d, p)$*;*
- **HS** *G has two minimal normal subgroups, which are nonabelian simple;*
- **HC** *G has two minimal normal subgroups, which are nonabelian non-simple;*
- **AS** *G is almost simple;*
- **SD** $N = \mathrm{soc}(G) = S^d$ *is minimal normal, and* $N_v \cong S$*;*
- **CD** $N = \mathrm{soc}(G) = S^d$ *is minimal normal, and* $N_v \cong S^e \neq S$*;*
- **TW** $\mathrm{soc}(G)$ *is minimal normal and regular;*
- **PA** $\mathrm{soc}(G)$ *is minimal normal, non-simple, and irregular.*

Thin edge-transitive graphs

We next consider locally-quasiprimitive graphs. We first exclude a trivial case.

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A thin edge-transitive graph is one of the following graphs:

- *a star;*
- *a cycle;*
- *the subdivision of a thick arc-transitive graph.*

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- *a cycle;*
- *the subdivision of a thick arc-transitive graph.*

We only need to consider thick graphs.

Locally-quasiprimitive graphs

Lemma.

If Γ is thick and G-locally-quasiprimitive, then a normal quotient Γ_N is (G/N)-locally-quasiprimitive, and either

- Γ_N *is a star, or*
- Γ_N *is thick.*

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Remarks:

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- *Γ_N is thick.*

Remarks:

- Cycles are not normal quotients of thick locally-quasiprimitive graphs.
- For locally-quasiprimitive graphs, Γ is not necessarily a cover of Γ_N, even if Γ_N is thick.

Example*

Let $R = \langle a_1, \ldots, a_6 \rangle \cong \mathbb{Z}_4^6$, and let $S = \{a_i, a_i^{-1} \mid 1 \leq i \leq 6\}$. Let $\Gamma = \mathrm{Cay}(R, S)$.

Then $\mathrm{Aut}\Gamma$ has a subgp $G = R{:}\mathrm{A}_5$ acting arc-transitively on Γ. As $G_v \cong \mathrm{A}_5$ is simple, Γ is G-locally-qp of valency 12.

As shown before, for the normal subgroup

$$N = \langle \hat{a}_i^2 \mid 1 \leq i \leq 6 \rangle \cong \mathbb{Z}_2^6,$$

Γ_N is of valency 6; in particular, Γ is not a cover of Γ_N.

Basic objects

Definition

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Lemma

If $\Gamma = (V, E)$ is basic G-locally-quasiprimitive and G is transitive on V, then G is quasiprimitive or bi-quasiprimitive on V.

(G being bi-quasiprimitive on V means that every minimal normal subgroup of G has exactly two orbits on V.)

Praeger (2003) gave a description of bi-quasiprimitive groups.

Basic locally-quasiprimitive graphs

Theorem.

Let $\Gamma = (V, E)$ *be a basic G-locally-quasiprimitive graph such that G is intrans on* V*. Then one of the following holds:*

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(i) $\Gamma = \mathbf{K}_{m,n}$, *and* $G = G_v G_w$, *where* (v, w) *is an edge;*

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Theorem.

Let $\Gamma = (V, E)$ be a basic G-locally-quasiprimitive graph such that G is intrans on V. Then one of the following holds:

- **(i)** $\Gamma = \mathbf{K}_{m,n}$, *and* $G = G_v G_w$, *where* (v, w) *is an edge;*
- **(ii)** *G is faithful and quasiprimitive on both biparts of type* $\{\mathrm{X}, \mathrm{Y}\}$, *where either* $\mathrm{X} = \mathrm{Y}$ *or* $\{\mathrm{X}, \mathrm{Y}\} = \{\mathrm{SD}, \mathrm{CD}\}$, $\{\mathrm{SD}, \mathrm{PA}\}$ *or* $\{\mathrm{CD}, \mathrm{PA}\}$. *Further,* Γ *is not locally-primitive for* $\{\mathrm{X}, \mathrm{Y}\} = \{\mathrm{SD}, \mathrm{CD}\}$; Γ *is not locally* 2*-arc-transitive for* $\{\mathrm{X}, \mathrm{Y}\} = \{\mathrm{CD}, \mathrm{PA}\}$.

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- **(iii)** *G is faithful on both biparts and quasiprimitive on exactly one of them, of which the quasiprimitive type is* HA, HS, HC, AS, PA *or* TW. *Moreover,* HC *type is not locally 2-arc-transitive.*

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(ii) G *is faithful and quasiprimitive on both biparts of type* $\{\mathrm{X}, \mathrm{Y}\}$*, where either* $\mathrm{X} = \mathrm{Y}$ *or* $\{\mathrm{X}, \mathrm{Y}\} = \{\mathrm{SD}, \mathrm{CD}\}$*,* $\{\mathrm{SD}, \mathrm{PA}\}$ *or* $\{\mathrm{CD}, \mathrm{PA}\}$*. Further,* Γ *is not locally-primitive for* $\{\mathrm{X}, \mathrm{Y}\} = \{\mathrm{SD}, \mathrm{CD}\}$*;* Γ *is not locally* 2*-arc-transitive for* $\{\mathrm{X}, \mathrm{Y}\} = \{\mathrm{CD}, \mathrm{PA}\}$*.*

(iii) G *is faithful on both biparts and quasiprimitive on exactly one of them, of which the quasiprimitive type is* HA*,* HS*,* HC*,* AS*,* PA *or* TW*. Moreover,* HC *type is not locally* 2*-arc-transitive.*

Examples exist for each of these cases.

Problems

This theorem suggests us to study the following problems.

Problem A

Characterise the graphs for each case appeared in the theorem.

Problem B

Find normal covers or multi-covers of basic locally-quasiprimitive graphs.

Dual actions

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Now the vertex set V of Γ is T, and also V can be identified with $[G : G_v]$ such that G acts on

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If $T \lhd G$, then Γ is a normal Cayley graph. Otherwise, T is core-free in G, and so G has another coset action on

$$\Omega = [G : T],$$

called a dual action of G on $[G : G_v]$.

Note that the stabiliser G_v is transitive on $[G : T]$.

By Tutte's theorem, $G_v = \mathbb{Z}_3, \mathrm{S}_3, \mathrm{D}_{12}, \mathrm{S}_4$ or $\mathrm{S}_4 \times \mathrm{S}_2$. Thus, the size of $[G : T]$ divides 48, and $G \leq \mathrm{S}_{48}$.

Proposition.

If Γ is not a normal Cayley graph, then T is one of the following groups:

$$\mathrm{A}_5, \mathrm{PSL}(2, 11), \mathrm{M}_{11}, \mathrm{A}_{11}, \mathrm{M}_{23}, \mathrm{A}_{23}, \text{ or } \mathrm{A}_{47}.$$

Shangjin Xu (PhD project, 2006) completed this work. In particular, for $T = \mathrm{A}_{47}$, two 5-arc-transitive graphs are found.

Constructing non-Cayley graphs

Let T be simple, and $T = \langle a, b \rangle$ with $o(a) = 3$, $o(b) = 2$. Let

$$\Gamma = \mathrm{Cos}(T, \langle a \rangle, \langle a \rangle b \langle a \rangle).$$

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Then $G := \mathrm{Aut}\Gamma = TG_v$ such that $|G_v|$ divides 48.

Suppose that Γ is a Cayley graph of R. Then $G = RG_v$. Let C be the core of R in G. Then $\overline{G} = \overline{R}\,\overline{G}_v$ such that $\overline{R}$ is core-free in $\overline{G}$ and $\overline{G}_v \cong G_v$. It follows that $T \cong \overline{T} \leq \overline{G} \leq \mathrm{S}_{48}$.

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Proposition.

If $T \not\leq \mathrm{S}_{48}$, then Γ is not a Cayley graph.

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Proposition.

If $T \not\leq \mathrm{S}_{48}$, then Γ is not a Cayley graph.

Theorem (Liebeck-Shalev)

Except for $\mathrm{Sz}(q)$, $\mathrm{Sp}(4, 2^f)$ and finitely many exceptions, every simple group is generated by an element of order 3 and an involution.

Vertex-trans graphs and Cayley graphs

Conjecture (McKay-Praeger 1990's)

Almost all vertex-transitive graphs are Cayley graphs.

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Theorem

Almost all 3-arc-trans graphs are not Cayley graphs.

Let $\Gamma = (V, E)$ be a connected $(G, 3)$-arc-transitive graph of valency k. We need a big theorem.

Theorem (Weiss, Trofimov)

The stabiliser G_v has order $|G_v|$ upper-bounded by a function $f(k)$.

Suppose that Γ is a Cayley graph of R. Then $G = RG_v$.

Let C be the core of R in G. Then C has ≥ 3 orbits on V.

Let $\overline{R} = R/C$, and $\overline{G} = G/C$.

Then $\overline{R}$ is core-free in $\overline{G}$, $\overline{G} = \overline{R}\,\overline{G}_v$, and $\overline{G}_v \cong G_v$.

Thus, $\overline{G} \leq \mathrm{Sym}(\Omega)$, where $\Omega = [\overline{G} : \overline{R}]$.

Now, $\overline{G}_v$ is transitive on Ω, and so $|\Omega|$ divides $|G_v|$. In particular,

$$|\Omega| \leq f(k), \text{ and } \overline{G} \leq \mathrm{Sym}(f(k)).$$

So for each k, only finitely many basic 3-arc-transitive graphs of valency k and their covers are Cayley graphs.