Two-fold orbital graphs and digraphs
Part 1

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Convention

An oriented graph is considered to be a finite set of vertices and a set or *ordered pairs* of vertices. If the arcs (x, y) and (y, x) both exist then we say that the arcs are *self-paired*. Together, a pair of self-paired arcs are considered to form the *edge* $\{a, b\}$.
Multiple arcs (repetition of the arc (x, y)) are not allowed, but loops (the arc (x, x)) are possible.

Convention

We distinguish two special types of oriented graphs.

If there is no loop (x, x) and *no* arc is self-paired then the oriented graph is said to be a *digraph*.

If there is no loop and *every* arc is self paired then we get a *graph*.

If the oriented graph is neither a graph or a digraph then we often call it a *mixed graph*.

Orbital Graphs

Let Γ be a permutation group acting transitively on a set V. Fix $(u, v) \in V \times V$. Then all pairs $(\alpha(u), \alpha(v))$, with $\alpha \in \Gamma$, form an oriented graph G such that $\Gamma \subseteq \mathrm{Aut}(G)$.

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If G is disconnected then all its components are isomorphic

Notation

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$\mathbf{\Gamma}$ will denote a subgroup of $\mathcal{S}$.

Suppose $\pi_1, \pi_2 : \mathbf{\Gamma} \to S_n$ are defined by $\pi_1((\alpha, \beta)) = \alpha$ and $\pi_2((\alpha, \beta)) = \beta$. Then π_1 and π_2 are said to be the *projections* of $\mathbf{\Gamma}$ on S_n.

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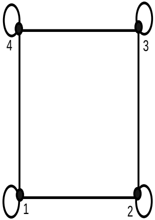
The oriented graph $G = (V, \mathbf{\Gamma}(u, v))$ is said to be a *two-fold orbital digraph (TOD)* or a *two-fold orbital graph (TOG)* if it is a digraph or a graph, respectively.

An example



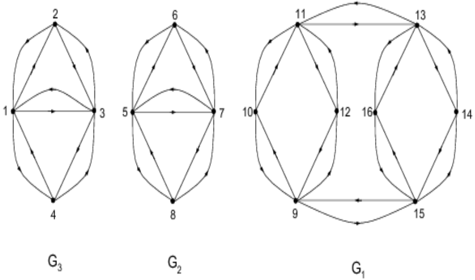
$$\Gamma = D_4 \times S_4 \leq S_4 \times S_4.$$

The graph G has arc-set $\Gamma(1,2)$.

The arc-set of G is self-paired although Γ is not.

The components of a disconnected TOG are not necessarily isomorphic

Example



TF-isomorphisms

Let G_1 and G_2 be two graphs or digraphs or mixed graphs. Then (α, β) is a *two-fold isomorphism* from G_1 to G_2 (*TF-isomorphism*) iff α and β are bijections from $V(G_1)$ to $V(G_2)$ such that (u, v) is an arc in G_1 iff $(\alpha(u), \beta(v))$ is an arc in G_2.

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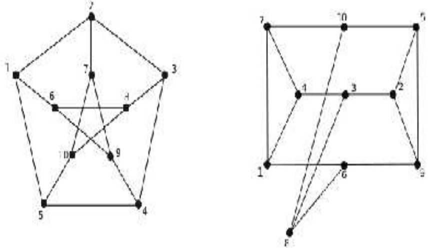
If $\alpha \neq \beta$ then (α, β) is said to be a *non-trivial* TF-isomorphism (or TF-automorphism).

An example



If $\alpha = (19)(24)(57)(3)(6)(8)(10)$ and β the identity then (α, β) is a TF-automorphism from the Petersen graph to the second graph shown in the figure.

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- Let G be a graph. Then if (α, β) is a two-fold automorphism, (id, β) is also a two fold automorphism iff (α, id) is a two-fold automorphism.

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- Let G be a graph. Then if (α, β) is a two-fold automorphism, (id, β) is also a two fold automorphism iff (α, id) is a two-fold automorphism.
- If (α, β) is a two-fold automorphism of a graph G such that α and β are of a different order, then there exists a non-trivial automorphism of G.

Canonical Double Covers

Let G be a digraph / graph / mixed graph. The *canonical double cover (CDC)* of G is the digraph $B(G)$ whose vertex-set is $V(G) \times \mathbb{Z}_2$ such that there exists an arc joining (u, ϵ) to $(v, \epsilon + 1)$ in $B(G)$ iff there exists an arc joining u to v in G.

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$B(G)$ is bipartite.

If G is bipartite then $B(G)$ is disconnected.

What is preserved by a TF-isomorphism?

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Theorem

Suppose G_1 and G_2 are either both graphs or both digraphs. Then $B(G_1) \simeq B(G_2)$, iff they are TF-isomorphic.

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If, in the above result, G_1 and G_2 are not both graphs or both digraphs, then there can be a TF-isomorphism between them but their CDC's are not isomorphic.

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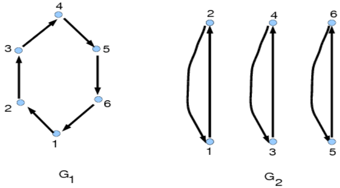
Example



Disconnected two-fold orbital graphs

Theorem

Let G be a TOG with no isolated vertices and let its connected components be $G_1, \ldots, G_k$ such that:

$$|V(G_1)| \geq |V(G_2)| \geq \ldots \geq |V(G_k)|.$$

Then each G_i is also a TOG. Moreover,

1. *if $|V(G_1)| = |V(G_k)|$, then $G_1, \ldots, G_k$ are pairwise TF-isomorphic;*
2. *otherwise, there exists a unique index $r \in \{1, \ldots, k-1\}$ such that*
    1. $$G_1 \simeq G_2 \simeq \ldots \simeq G_r;$$
    2. *none of $G_{r+1}, \ldots, G_k$ is isomorphic or TF-isomorphic to G_1;*
    3. $G_{r+1} \simeq^{TF} \ldots \simeq^{TF} G_k$; *and*
    4. $G_1 \simeq B(G_k)$

An example

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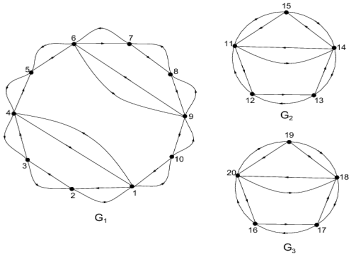

Figure: G_2 and G_3 are (TF-)isomorphic and G_1 is a CDC of each.

Bipartite disconnected TOGs

Theorem

Let G be a disconnected TOG with no isloated vertices, and let its connected components be $G_1, \ldots, G_k$. If one of the components is bipartite then:

1. *either all components are isomorphic; or*
2. *there exists a unique index $r \in \{1, \ldots, k-1\}$ such that:*
   1. $$G_1 \simeq G_2 \simeq \ldots \simeq G_r;$$
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   3. $G_{r+1} \simeq^{TF} \ldots \simeq^{TF} G_k$;
   4. $G_1 \simeq B(G_k)$; *and*
   5. *all $G_1, \ldots, G_r$ are bipartite but no $G_{r+1}, \ldots, G_k$ is bipartite.*

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Corollary. *A bipartite disconnected TOG has all of its components isomorphic*

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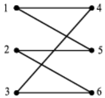

Figure: The graph C_6 has a non-trivial TF-automorphism (α, β) with $\alpha = (123)(4)(5)(6)$ and $\beta = (1)(2)(3)(456)$. Neither α nor β is an element of the automorphism group of G.

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However...

Thank you!