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Arc-transitive elementary-abelian covers of graphs

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Regular covering projections

$X \equiv$ a finite, simple, connected and undirected graph

Covering projection: a locally bijective graph epimorphism

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(locally bijective: restrictions $N(\tilde{v}) \to N(v)$ are bijective)

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Isomorphic covering projections: for some $\alpha \in \mathrm{Aut}(X)$, we have

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}} & \tilde{X}' \\ \downarrow q & & \downarrow q' \\ X & \xrightarrow{\alpha} & X \end{array}$$

In particular, q and q' are equivalent if $\alpha = \mathrm{id}_X$.

Regular covering projections

Regular covering projection: $\exists H \leq \mathrm{Aut}(\tilde{X})$ semiregular s. t.

$$\tilde{X}/H \cong X$$

(that is, H-orbits of $\tilde{X}$ are the vertex fibres $q^{-1}(v)$, $v \in V(X)$).
We call $H = \mathrm{CT}(q)$ the group of covering transformations.
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Theorem (Gross, Tucker, 1973)

Any regular $q\colon \tilde{X} \to X$ is equivalent to some

$$q_\zeta\colon X \times_\zeta H \to X,\ q_\zeta(u,h) = u,$$

where voltage assignment $\zeta\colon A(X) \to H$ satisfies $\zeta(u,v) = (\zeta(v,u))^{-1}$ and the derived covering graph $X \times_\zeta H$ has vertex set $V(X) \times H$ and edges defined by

$$u \sim v \iff (u,g) \sim (v, g\zeta(u,v)).$$

Example

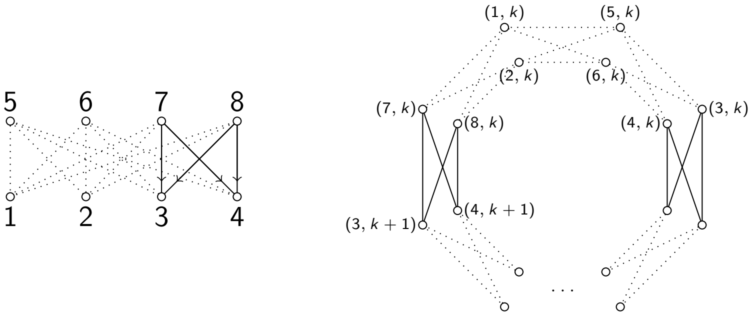

Figure: $K_{4,4} \times_{\zeta} \mathbb{Z}_p$, where $\zeta(a) = 1$ on denoted arcs and trivial elsewhere.

$\alpha \in \mathrm{Aut}(X)$ lifts along q if

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}} & \tilde{X} \\ \downarrow q & & \downarrow q \\ X & \xrightarrow{\alpha} & X \end{array}$$

commutes for some $\tilde{\alpha} \in \mathrm{Aut}(\tilde{X})$. For $G \leq \mathrm{Aut}(X)$, covering projection q is G-admissible if each $\alpha \in G$ lifts.

Lifting automorphisms

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Lemma (Djoković, 1974)

$\tilde{X}$ connected. If q is G-admissible for some VT/ET/s-AT group $G \leq \mathrm{Aut}(X)$, then $\tilde{G}$ (and hence $\tilde{X}$) is VT/ET/s-AT.

Corollary

There are infinitely many finite connected cubic 5-AT graphs.

Lifts using voltages

Since then, many applications in graph theory appeared:

- Constructions of examples with particular symmetry properties.
- Computer-assisted compilation of lists.
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Malnič, 1996: covers of generalized graphs with *semiedges*.

Theorem (Malnič, Nedela, Škoviera, 2000: Basic lifting lemma for regular covers)

$\alpha \in \mathrm{Aut}(X)$ *lifts along a regular covering projection* q $\iff$ $\zeta(W) = 1 \implies \zeta(\alpha W) = 1$ *for each closed walk* $W \in \pi_1(X, v)$.

Concrete questions remain hard to answer:

- Given $p\colon \tilde{X} \to X$, find all $\alpha \in \mathrm{Aut}(X)$ that lift.
- Given X and $G \leq \mathrm{Aut}(X)$, find all G-admissible q.

Elementary-abelian covering projections

If $\mathrm{CT}(q) = \mathbb{Z}_p^k$, the covering projection q is elementary-abelian.

In this case, there is a natural linear representation of $\mathrm{Aut}(X)$:

$$[\quad]\colon \mathrm{Aut}(X) \hookrightarrow \mathrm{GL}(r, \mathbb{Z}_p), \alpha \mapsto [\alpha] \in M_r(\mathbb{Z}_p)$$

(By acting on directed cycles of X, each $\alpha \in \mathrm{Aut}(X)$ induces a linear mapping on $H_1(X, \mathbb{Z}_p) \cong \mathbb{Z}_p^{r\times 1}$.)

Theorem (Malnič, Marušič, Potočnik, 2003)

G-admissible elementary-abelian covering projections of X correspond to $[G]^t$-invariant subspaces of $\mathbb{Z}_p^{r\times 1}$. Moreover,

- *Choice of basis is invariant up to equivalence of projections.*
- *Voltage assignments are obtained from inv. subspace basis.*
- *q_U, q_V isomorphic $\iff$ $[\alpha]^t U = V$ for some $\alpha \in \mathrm{Aut}(X)$.*

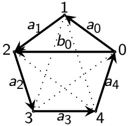
Notes

Du, Kwak, Xu, 2003: Alternative method for lifting α along elementary-abelian q.

Theorem [MMP] generalizes observations by Širan (2001): For $\mathrm{CT}(q) = \mathbb{Z}_p$, lifting $\alpha \in \mathrm{Aut}(X)$ is related to eigenvectors of $[\alpha]$.

Theorem (Some nontrivial results obtained by MMP method)

- *SS EAC of the Heawood graph (Malnič, Marušič, Potočnik)*
- *VT EAC of the Petersen graph (Malnič, Potočnik)*
- *SS EAC of the Moebius-Kantor graph (Malnič, Marušič, Miklavič, Potočnik)*
- *AT EAC of the Octahedron graph (Kwak, Oh)*
- *AT EAC of the Pappus and the Dodecahedron Graph (Oh)*
- *AT EAC of graphs K_5 and $K_{4,4}$ (K.)*



$\mathrm{Aut}(K_5) = S_5 = \langle \rho, \tau, \sigma \rangle$,
where $\rho = (01234)$, $\tau = (0132)$, $\sigma = (024)$.
Minimal AT subgroups are $\langle \rho, \tau \rangle$ and $\langle \rho, \sigma \rangle$.

#	Voltage assignment ζ						G-admissible	Conditions
	$\zeta(c_0)$	$\zeta(c_1)$	$\zeta(c_2)$	$\zeta(c_3)$	$\zeta(c_4)$	$\zeta(c)$	for AT subgroup	
1.	$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$	$\langle \rho, \sigma, \tau \rangle$	p any prime
2.	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\langle \rho, \tau \rangle$	
3.	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\langle \rho, \tau \rangle$	
4.	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}1\end{pmatrix}$	$\begin{pmatrix}2+\iota\end{pmatrix}$	$\langle \rho, \tau \rangle$	$p \neq 5$ $p = 1 \pmod 4$ $\iota^2 = -1 \pmod p$
5.	$\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\\-1\\-1\\-1\end{pmatrix}$	$\begin{pmatrix}2+\iota\\0\\0\\0\\0\end{pmatrix}$	$\langle \rho, \tau \rangle$	
6.	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\\\eta\end{pmatrix}$	$\begin{pmatrix}1\\-\eta\\-\eta\end{pmatrix}$	$\begin{pmatrix}1\\\eta\\-1\end{pmatrix}$	$\begin{pmatrix}1+2\eta\\0\\0\end{pmatrix}$	$\langle \rho, \sigma \rangle$	$p = \pm 1 \pmod 5$, $\eta^2 + \eta = 1 \pmod p$
7.	2 sporadic cases							p=2
8.	12 sporadic cases							p=5

Notes

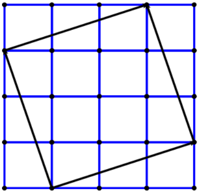
- A homology basis reflecting the rotational symmetry of K_5 was deliberately chosen in order to simplify the computations.
- Subcases for different p essentialy depend on the factorization of the minimal polynomials of respective matrices over $\mathbb{Z}_p$.
- For p not dividing $|G|$, a classical theorem by Maschke (1892?) can be applied in order to construct all $[G]$-invariant subspaces as direct sums of minimal ones.
- Voltage functions carry plenty of information on respective covers. For instance, covering graphs with $\mathrm{girth}(X) \leq 5$ are easily identified.

Example

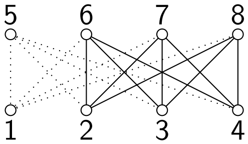
Potočnik, Wilson, 2006: Classification of ET 4-valent graphs of girth at most 4.

Corollary

Let $\tilde{X}$ be an arc-transitive cover of K_5. Then: $\mathrm{Girth}(\tilde{X}) = 4$ $\iff$ $\tilde{X}$ *admits a toroidal embedding (of type* $\{4,4\}_{a,b}$*).*



Example: Minimal nontrivial AT EAC of K_5 corresponds to toroidal map of type $\{4,4\}_{3,1}$.



- $\mathrm{Aut}(X)$ is relatively large and complicated.
- There are (up to conjugacy) 6 minimal AT subgroups, each generated by some subset of 8 different automorphisms:

$$\langle s, r, a \rangle, \langle s, r, b \rangle, \langle s, r, c \rangle, \langle s, r, d \rangle, \langle t, d \rangle, \langle t, e \rangle.$$

- The problem essentially reduces to simultaneous block-diagonalization of certain 9×9 matrices.

Arc-transitive elementary-abelian covers of $K_{4,4}$

- Maschke's theorem applies for all $p \neq 2$. However, it is hard to identify all minimal invariant subspaces.
- Instead, if some G-invariant subspace is known, then all its G-invariant complements appear as solutions of a certain linear system.
- With a couple of tricks, large G-invariant subspaces are split to smaller ones or proven minimal (independently of $p \neq 2$).
- The resulting minimal non-equivalent AT covering projections are further reduced to 8 different isomorphism classes.

Thank you!