Regular Cayley maps over dihedral groups D_{2n}, n is an odd number

István Kovács[a], Dragan Marušič[a,b], and Mikhail E. Muzychuk[c]

[a] *FAMNIT, University of Primorska, Slovenia*

[b] *PeF, University of Ljubljana, Slovenia*

[c] *Department of Computer Science and Mathematics, Netanya Academic College, Israel*

August 5, 2010, Rogla, Slovenia

Maps

Definition. A map $\mathcal{M}$ with an underlying connected graph Γ is a triple

$$\mathcal{M} = (\Gamma; R, L)$$

where R (rotation) is a permutation of the arc set $A(\Gamma)$ whose orbits are the sets of arcs initiated from the same vertex, and L (dart-reversing involution) is an involution of $A(\Gamma)$ whose orbits are the sets of arcs based on the same edge.

Regular maps

Definition. An isomorphism $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a bijection $\varphi\colon A(\Gamma_1) \to A(\Gamma_2)$ such that

$$R_1\varphi = \varphi R_2 \text{ and } L_1\varphi = \varphi L_2.$$

Regular maps

Definition. An isomorphism $\varphi \colon \mathcal{M}_1 \to \mathcal{M}_2$ is a bijection $\varphi \colon A(\Gamma_1) \to A(\Gamma_2)$ such that

$$R_1\varphi = \varphi R_2 \text{ and } L_1\varphi = \varphi L_2.$$

Definition. An automorphism of a map $\mathcal{M} = (\Gamma; R, L)$ is an automorphism φ of Γ whose action on $A(\Gamma)$ is an automorphism of $\mathcal{M}$.

Regular maps

Definition. An isomorphism $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a bijection $\varphi\colon A(\Gamma_1) \to A(\Gamma_2)$ such that

$$R_1\varphi = \varphi R_2 \text{ and } L_1\varphi = \varphi L_2.$$

Definition. An automorphism of a map $\mathcal{M} = (\Gamma; R, L)$ is an automorphism φ of Γ whose action on $A(\Gamma)$ is an automorphism of $\mathcal{M}$.

Fact. For a map $\mathcal{M} = (\Gamma; R, L)$ the automorphism group $\mathrm{Aut}(\mathcal{M})$ acts semiregularly on $A(\Gamma)$, therefore $|\,\mathrm{Aut}(\mathcal{M})| \leq |A(\Gamma)|$.

Regular maps

Definition. An isomorphism $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a bijection $\varphi\colon A(\Gamma_1) \to A(\Gamma_2)$ such that

$$R_1\varphi = \varphi R_2 \text{ and } L_1\varphi = \varphi L_2.$$

Definition. An automorphism of a map $\mathcal{M} = (\Gamma; R, L)$ is an automorphism φ of Γ whose action on $A(\Gamma)$ is an automorphism of $\mathcal{M}$.

Fact. For a map $\mathcal{M} = (\Gamma; R, L)$ the automorphism group $\mathrm{Aut}(\mathcal{M})$ acts semiregularly on $A(\Gamma)$, therefore $|\,\mathrm{Aut}(\mathcal{M})| \leq |A(\Gamma)|$.

Definition. A map $\mathcal{M}$ is regular if $|\,\mathrm{Aut}(\mathcal{M})| = |A(\Gamma)|$.

Cayley maps

Definition. Let S be a subset of a group G, $1 \notin S$, $S = S^{-1}$, and $\langle S \rangle = G$. The Cayley graph $\mathrm{Cay}(G, S)$ with connection set S is the graph (V, E) such that

$$V = G \text{ and } E = \left\{ \{x, sx\} \mid x \in G,\ s \in S \right\}.$$

Cayley maps

Definition. Let S be a subset of a group G, $1 \notin S$, $S = S^{-1}$, and $\langle S \rangle = G$. The Cayley graph $\mathrm{Cay}(G, S)$ with connection set S is the graph (V, E) such that

$$V = G \text{ and } E = \{\, \{x, sx\} \mid x \in G,\, s \in S \,\}.$$

Definition. Let S be a subset of a group G, $1 \notin S$, $S = S^{-1}$, $\langle S \rangle = G$, and let p be a cyclic permutation of S. The Cayley map $\mathrm{CM}(G, S, p)$ is the map $\mathcal{M} = (\Gamma; R, L)$, with $\Gamma = \mathrm{Cay}(G, S)$, and the rotation R is defined as

$$(x, sx)^R = (x, p(s)x),\ x \in G,\ s \in S.$$

Cayley maps

Definition. Let S be a subset of a group G, $1 \notin S$, $S = S^{-1}$, and $\langle S \rangle = G$. The Cayley graph $\mathrm{Cay}(G, S)$ with connection set S is the graph (V, E) such that

$$V = G \text{ and } E = \{\, \{x, sx\} \mid x \in G,\, s \in S \,\}.$$

Definition. Let S be a subset of a group G, $1 \notin S$, $S = S^{-1}$, $\langle S \rangle = G$, and let p be a cyclic permutation of S. The Cayley map $\mathrm{CM}(G, S, p)$ is the map $\mathcal{M} = (\Gamma; R, L)$, with $\Gamma = \mathrm{Cay}(G, S)$, and the rotation R is defined as

$$(x, sx)^R = (x, p(s)x),\ x \in G,\ s \in S.$$

Fact. If $\mathcal{M}$ is a Cayley map $\mathcal{M}$ over a group G, then the right regular representation $G_* \leq \mathrm{Aut}(\mathcal{M})$ ($g_* \in G_*$ acts as $x^{g_*} = xg$, $x \in G$).

Regular Cayley maps and skew-morphisms

Definition. A skew-morphism of a group G is a permutation ψ of G such that there exists a function $\pi\colon G \to \{0, 1, \ldots, m-1\}$ (power function), where m is the order of ψ, such that

- $1^\psi = 1$,
- $(xy)^\psi = x^{\psi^{\pi(y)}} y^\psi$ for all $x, y \in K$.

Regular Cayley maps and skew-morphisms

Definition. A skew-morphism of a group G is a permutation ψ of G such that there exists a function $\pi\colon G \to \{0, 1, \ldots, m-1\}$ (power function), where m is the order of ψ, such that

- $1^\psi = 1$,
- $(xy)^\psi = x^{\psi^{\pi(y)}} y^\psi$ for all $x, y \in K$.

Theorem. (Jajcay, Širáň 2002) A Cayley map $\mathrm{CM}(G, S, p)$ is regular if and only if there exists a skew-morphism of G such that S is an orbit of G and $\psi|_S = p$.

Regular Cayley maps and skew-morphisms

Definition. A skew-morphism of a group G is a permutation ψ of G such that there exists a function $\pi\colon G \to \{0, 1, \dots, m-1\}$ (power function), where m is the order of ψ, such that

- $1^\psi = 1$,
- $(xy)^\psi = x^{\psi^{\pi(y)}} y^\psi$ for all $x, y \in K$.

Theorem. (Jajcay, Širáň 2002) A Cayley map $\mathrm{CM}(G, S, p)$ is regular if and only if there exists a skew-morphism of G such that S is an orbit of G and $\psi|_S = p$.

Proposition. A permutation ψ of G is skew-morphisms if and only if $1^\psi = 1$ and $X = \langle G_*, \psi\rangle$ has stabilizer $X_1 = \langle\psi\rangle$.

t-balanced Cayley maps

Definition. A Cayley map $\mathcal{M} = \mathrm{CM}(G, S, p)$ is t-balanced if $p(s)^{-1} = p^t(s^{-1})$ for all $s \in S$. In particular, if $t = 1$, then we say that $\mathcal{M}$ is balanced.

t-balanced Cayley maps

Definition. A Cayley map $\mathcal{M} = \mathrm{CM}(G, S, p)$ is t-balanced if $p(s)^{-1} = p^t(s^{-1})$ for all $s \in S$. In particular, if $t = 1$, then we say that $\mathcal{M}$ is balanced.

Theorem. (Conder, Jajcay and Tucker) Let $\mathcal{M} = \mathrm{CM}(G, S, p)$ be a regular Cayley map with associated skew-morphism ψ and power-function π.

- $\mathcal{M}$ is balanced if and only if ψ is an automorphism of G.
- $\mathcal{M}$ is t-balanced for $t > 1$ if and only if $t^2 \equiv 1 \pmod{|S|}$, π has only two values 1 and t,

$$\{x \in G \mid \pi(x) = 1\} = G^+ \leq G,$$

  such that $[G\colon G^+] = 2$, ψ fixes G^+, and $\psi|_{G^+}$ is an automorphism of G^+.

Classification results

The only class of finite groups over which all regular Cayley maps are classified is the class of cyclic groups due to Conder and Tucker.

Classification results

The only class of finite groups over which all regular Cayley maps are classified is the class of cyclic groups due to Conder and Tucker.

Besides this result only partial classifications are known:

- regular balanced Cayley maps over dihedral and generalized quaternion groups (Wang and Feng, 2005).
- regular t-balanced Cayley maps over dihedral groups (Kwak, Kwon, and Feng, 2006),
- regular t-balanced Cayley maps over dicyclic groups (Kwak and Oh, 2008)
- regular t-balanced Cayley maps over semi-dihedral groups (Oh, 2009).

Classification results

The only class of finite groups over which all regular Cayley maps are classified is the class of cyclic groups due to Conder and Tucker.

Besides this result only partial classifications are known:

- regular balanced Cayley maps over dihedral and generalized quaternion groups (Wang and Feng, 2005).
- regular t-balanced Cayley maps over dihedral groups (Kwak, Kwon, and Feng, 2006),
- regular t-balanced Cayley maps over dicyclic groups (Kwak and Oh, 2008)
- regular t-balanced Cayley maps over semi-dihedral groups (Oh, 2009).

Remark. To classify regular Cayley maps over dihedral groups one needs to consider only those maps that are not t-balanced for any t.

Construction of non t-balanced Cayley maps over D_{2n}, n is odd

$$D_{2n} = \langle r, s \mid r^2 = s^2 = (rs)^n = 1 \rangle, c = rs, C_n = \langle c \rangle.$$

Construction of non t-balanced Cayley maps over D_{2n}, n is odd

$$D_{2n} = \langle r, s \mid r^2 = s^2 = (rs)^n = 1 \rangle, c = rs, C_n = \langle c \rangle.$$

Let $\mathfrak{T}$ be the set of all pairs (n, ℓ) of positive integers satisfying the following conditions:

- n is an odd number, $n \equiv 0 \pmod 3$, and
- ℓ is an element in $\mathbb{Z}_n^*$ of odd order m.

Construction of non t-balanced Cayley maps over D_{2n}, n is odd

$$D_{2n} = \langle r, s \mid r^2 = s^2 = (rs)^n = 1\rangle, c = rs, C_n = \langle c\rangle.$$

Let $\mathfrak{T}$ be the set of all pairs (n, ℓ) of positive integers satisfying the following conditions:

- n is an odd number, $n \equiv 0 \pmod 3$, and
- ℓ is an element in $\mathbb{Z}_n^*$ of odd order m.

For each (n, ℓ) in $\mathfrak{T}$ we define a Cayley map $\mathrm{CM}(D_{2n}, S, p)$, which we denote also by $\mathrm{CM}(n, \ell)$, as follows:

$$\begin{aligned} S &= \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \ldots, m-1\}\}, \\ p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}). \end{aligned}$$

Construction of non t-balanced Cayley maps over D_{2n}, n is odd

$$D_{2n} = \langle r, s \mid r^2 = s^2 = (rs)^n = 1 \rangle, c = rs, C_n = \langle c \rangle.$$

Let $\mathfrak{T}$ be the set of all pairs (n, ℓ) of positive integers satisfying the following conditions:

- n is an odd number, $n \equiv 0 \pmod 3$, and
- ℓ is an element in $\mathbb{Z}_n^*$ of odd order m.

For each (n, ℓ) in $\mathfrak{T}$ we define a Cayley map $\mathrm{CM}(D_{2n}, S, p)$, which we denote also by $\mathrm{CM}(n, \ell)$, as follows:

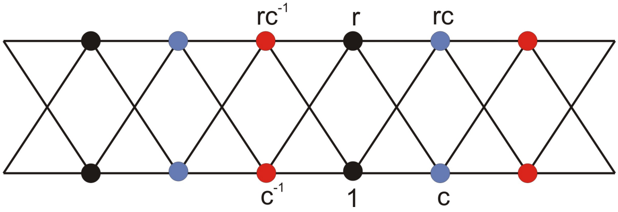
$$\begin{aligned} S &= \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \ldots, m-1\}\}, \\ p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}). \end{aligned}$$

Proposition. The Cayley map $\mathrm{CM}(n, \ell)$ is regular and not t-balanced for any t.

Proof. CM(n, ℓ) is regular:
Case $\ell = 1$. $S = \{c, c^{-1}, rc, rc^{-1}\}$, $p = (c, rc^{-1}, rc, c^{-1})$.

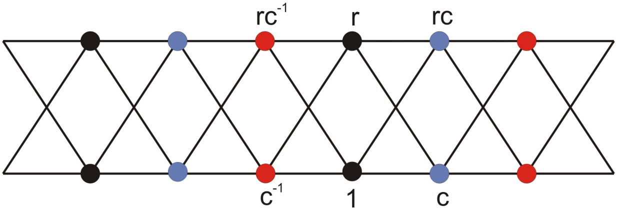
Proof. CM(n, ℓ) is regular:

Case $\ell = 1$. $S = \{c, c^{-1}, rc, rc^{-1}\}$, $p = (c, rc^{-1}, rc, c^{-1})$.



Proof. CM(n, ℓ) is regular:

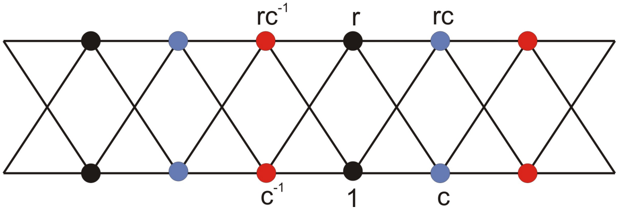
Case $\ell = 1$. $S = \{c, c^{-1}, rc, rc^{-1}\}$, $p = (c, rc^{-1}, rc, c^{-1})$.



$$N = \langle \mu_{\mathrm{blue}}, \mu_{\mathrm{red}} \rangle \cong \mathbb{Z}_2^2.$$

Proof. CM(n, ℓ) is regular:

Case $\ell = 1$. $S = \{c, c^{-1}, rc, rc^{-1}\}$, $p = (c, rc^{-1}, rc, c^{-1})$.

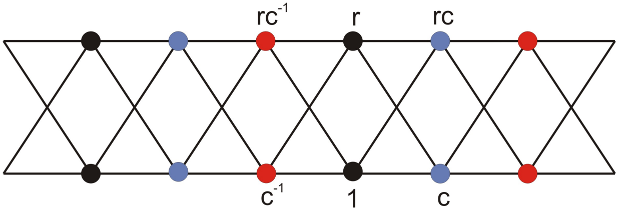


$$N = \langle \mu_{\mathrm{blue}}, \mu_{\mathrm{red}} \rangle \cong \mathbb{Z}_2^2.$$

N is normalized by D_*, $G = N \rtimes D_*$.

Proof. $\mathrm{CM}(n,\ell)$ is regular:

Case $\ell = 1$. $S = \{c, c^{-1}, rc, rc^{-1}\}$, $p = (c, rc^{-1}, rc, c^{-1})$.



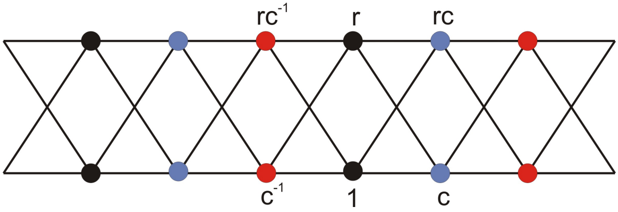
$$N = \langle \mu_{\mathrm{b}lue}, \mu_{\mathrm{r}ed} \rangle \cong \mathbb{Z}_2^2.$$

N is normalized by D_*, $G = N \rtimes D_*$.

$G_1 = \langle \mu_{\mathrm{b}lue} r_* \rangle = \langle \psi \rangle$, ψ is a skew-morphisms.

Proof. CM(n, ℓ) is regular:

Case $\ell = 1$. $S = \{c, c^{-1}, rc, rc^{-1}\}$, $p = (c, rc^{-1}, rc, c^{-1})$.



$$N = \langle \mu_{\mathrm{b}lue}, \mu_{\mathrm{r}ed} \rangle \cong \mathbb{Z}_2^2.$$

N is normalized by D_*, $G = N \rtimes D_*$.

$G_1 = \langle \mu_{\mathrm{b}lue} r_* \rangle = \langle \psi \rangle$, ψ is a skew-morphisms.

$\psi|_S = (c, c^{\psi}, c^{\psi^2}, c^{\psi^3}) = p$, CM($n$, 1) is regular.

General case:

$$S = \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \dots, m-1\}\},$$
$$p = (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}).$$

General case:

$$\begin{aligned} S &= \left\{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \ldots, m-1\}\right\}, \\ p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}). \end{aligned}$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

General case:

$$S = \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \ldots, m-1\}\},$$
$$p = (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}).$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

N is normalized by σ, $G = (N \rtimes D_*) \rtimes \langle\sigma\rangle$.

General case:

$$
\begin{aligned}
S &= \left\{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \dots, m-1\}\right\}, \\
p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}).
\end{aligned}
$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

N is normalized by σ, $G = (N \rtimes D_*) \rtimes \langle \sigma \rangle$.

$G_1 = \langle \mu_{\mathrm{b}lue} r_*, \sigma \rangle = \langle \psi \rangle$.

General case:

$$S = \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \dots, m-1\}\},$$
$$p = (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}).$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

N is normalized by σ, $G = (N \rtimes D_*) \rtimes \langle\sigma\rangle$.

$G_1 = \langle \mu_{\mathrm{b}lue} r_*, \sigma\rangle = \langle\psi\rangle$.

$\psi|_S = p$, $\mathrm{CM}(n, \ell)$ is regular.

General case:

$$\begin{aligned} S &= \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0,\ldots,m-1\}\}, \\ p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}). \end{aligned}$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

N is normalized by σ, $G = (N \rtimes D_*) \rtimes \langle\sigma\rangle$.

$G_1 = \langle \mu_{\mathrm{b}lue} r_*, \sigma\rangle = \langle\psi\rangle$.

$\psi|_S = p$, $\mathrm{CM}(n,\ell)$ is regular.

$\mathrm{CM}(n,\ell)$ is not t-balanced:

General case:

$$\begin{aligned} S &= \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0,\ldots,m-1\}\}, \\ p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}). \end{aligned}$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

N is normalized by σ, $G = (N \rtimes D_*) \rtimes \langle\sigma\rangle$.

$G_1 = \langle \mu_{blue} r_*, \sigma\rangle = \langle\psi\rangle$.

$\psi|_S = p$, $\mathrm{CM}(n,\ell)$ is regular.

$\mathrm{CM}(n,\ell)$ is not t-balanced:

- $c^\psi = rc^{-1}$, $\psi \notin \mathrm{Aut}(D_{2n})$, not balanced.

General case:

$$\begin{aligned} S &= \{c^{\ell^i}, c^{-\ell^i}, rc^{\ell^i}, rc^{-\ell^i} \mid i \in \{0, \ldots, m-1\}\}, \\ p &= (c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}). \end{aligned}$$

Take σ in $\mathrm{Aut}(D_{2n})$ defined by $r^\sigma = r$ and $c^\sigma = c^\ell$.

N is normalized by σ, $G = (N \rtimes D_*) \rtimes \langle\sigma\rangle$.

$G_1 = \langle \mu_{\mathrm{blue}} r_*, \sigma \rangle = \langle \psi \rangle$.

$\psi|_S = p$, $\mathrm{CM}(n, \ell)$ is regular.

$\mathrm{CM}(n, \ell)$ is not t-balanced:

- $c^\psi = rc^{-1}$, $\psi \notin \mathrm{Aut}(D_{2n})$, not balanced.
- ψ does not fix $C_n < D_{2n}$, not t-balanced for any $t > 1$.

□

Classification of regular Cayley maps over D_{2n}, n is odd

Theorem (K, Marušič, Muzychuk) Let $\mathcal{M}$ be a regular Cayley map over a dihedral group D_{2n} such that n is odd and it is not t-balanced for any t.

(i) $\mathcal{M}$ is isomorphic to a map $\mathrm{CM}(n, \ell)$ for some $(n, \ell) \in \mathcal{T}$.

(ii) For any two pairs $(n, \ell_1), (n, \ell_2) \in \mathcal{T}$, the maps $\mathrm{CM}(n, \ell_1)$ and $\mathrm{CM}(n, \ell_2)$ are isomorphic if and only if $\ell_1 = \ell_2$.

.

Ingredients of proof I: arc-inverting involutions

Definition. A graph Γ is *G-arc-regular* if $G \leq \mathrm{Aut}(\Gamma)$, and G acts regularly on the arc set $A(\Gamma)$.

Ingredients of proof I: arc-inverting involutions

Definition. A graph Γ is G-arc-regular if $G \leq \mathrm{Aut}(\Gamma)$, and G acts regularly on the arc set $A(\Gamma)$.

Lemma. Let Γ be a connected and G-arc-regular graph. Then the following hold.

(i) For each arc (x, y) of Γ, there exists a unique involution $t_{xy} \in G$ which inverts (x, y) (that is, $x^{t_{xy}} = y$ and $y^{t_{xy}} = x$).

(ii) The set of all involutions $T = \{t_{xy} \mid (x, y) \in A(\Gamma)\}$ form a single conjugacy class of G.

(iii) For each $t \in T$, the centralizer $C_G(t)$ acts regularly on the set of all arcs inverted by t.

(iv) For each arc (x, y) of Γ, $G = \langle G_x, t_{xy} \rangle$.

Ingredients of proof I: arc-inverting involutions

Definition. A graph Γ is G-arc-regular if $G \leq \mathrm{Aut}(\Gamma)$, and G acts regularly on the arc set $A(\Gamma)$.

Lemma. Let Γ be a connected and G-arc-regular graph. Then the following hold.

(i) For each arc (x, y) of Γ, there exists a unique involution $t_{xy} \in G$ which inverts (x, y) (that is, $x^{t_{xy}} = y$ and $y^{t_{xy}} = x$).

(ii) The set of all involutions $T = \{t_{xy} \mid (x, y) \in A(\Gamma)\}$ form a single conjugacy class of G.

(iii) For each $t \in T$, the centralizer $C_G(t)$ acts regularly on the set of all arcs inverted by t.

(iv) For each arc (x, y) of Γ, $G = \langle G_x, t_{xy} \rangle$.

Remark. The involutions t_{xy} in the above lemma are called the arc-inverting involutions of Γ in G (Conder, Jajcay and Tucker 2007).

Lemma. Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected, G-arc-regular graph, and T be the set of arc-inverting involutions of Γ in G. Then the following hold.

(i) For each $s \in S \setminus C_n$, the permutation s_* is in T, and

$$|C_G(s_*)| = |C_{D_{2n}}(s)| \cdot |S \cap s^{D_{2n}}|,$$

where $s^{D_{2n}}$ is a conjugacy class of s in D_{2n}.

(ii) If $|S \cap C_n| = |S|/2$ and n is odd, then $|T| = 2n$.

Lemma. Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected, G-arc-regular graph, and T be the set of arc-inverting involutions of Γ in G. Then the following hold.

(i) For each $s \in S \setminus C_n$, the permutation s_* is in T, and

$$|C_G(s_*)| = |C_{D_{2n}}(s)| \cdot |S \cap s^{D_{2n}}|,$$

where $s^{D_{2n}}$ is a conjugacy class of s in D_{2n}.

(ii) If $|S \cap C_n| = |S|/2$ and n is odd, then $|T| = 2n$.

Corollary. Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be a regular Cayley map such that n is odd. Then $\mathcal{M}$ is balanced $\iff$ $S \cap C_n = \emptyset$.

Lemma. Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected, G-arc-regular graph, and T be the set of arc-inverting involutions of Γ in G. Then the following hold.

(i) For each $s \in S \setminus C_n$, the permutation s_* is in T, and

$$|C_G(s_*)| = |C_{D_{2n}}(s)| \cdot |S \cap s^{D_{2n}}|,$$

where $s^{D_{2n}}$ is a conjugacy class of s in D_{2n}.

(ii) If $|S \cap C_n| = |S|/2$ and n is odd, then $|T| = 2n$.

Corollary. Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be a regular Cayley map such that n is odd. Then $\mathcal{M}$ is balanced $\iff$ $S \cap C_n = \emptyset$.

Proof. $\Rightarrow$ Because of $\langle S \rangle = G$ and all $s \in S$ are of the same order.
$\Leftarrow$ $\mathrm{Cay}(D_{2n}, S)$ is G-arc-regular, $G = \mathrm{Aut}(\mathcal{M})$. The above lemma implies in turn that $|C_G(s_*)| = 2|S|$, $|s_*^G| = n = s^{D_*}$, $\langle s_*^G \rangle = D_* \trianglelefteq G$, $\mathcal{M}$ is balanced. $\square$

Ingredients of proof II: G-arc-regular dihedrants with trivial cyclic core

Definition. For a group A and its subgroup $B \leq A$, the core of B in A is the largest normal subgroup of A contained in B, notation: $core_A(B)$. The subgroup $B \leq A$ is core-free if $\mathrm{core}_A(B) = 1$.

Ingredients of proof II: G-arc-regular dihedrants with trivial cyclic core

Definition. For a group A and its subgroup $B \leq A$, the core of B in A is the largest normal subgroup of A contained in B, notation: $core_A(B)$. The subgroup $B \leq A$ is core-free if $\mathrm{core}_A(B) = 1$.

Theorem. (K, Marušič, and Muzychuk) Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected, G-arc-regular graph such that $(D_{2n})_* \leq G$, and $(C_n)_*$ is core-free in G. Then one of the following holds.

(i) $n = 1$, $\Gamma \cong K_2$, and $G \cong S_2$,

(ii) $n = 2$, $\Gamma \cong K_4$, and $G \cong A_4$,

(iii) $n = 3$, $\Gamma \cong K_{2,2,2}$, and $G \cong S_4$,

(iv) $n = 2m$, m is an odd number, $\Gamma \cong K_{n,n}$, and $G \cong (D_n \times D_n) \rtimes \langle \sigma \rangle$, where σ is an automorphism of $D_n \times D_n$ interchanging the coordinates (that is, $(x, y)^\sigma = (y, x)$ for all $(x, y) \in D_n \times D_n$).

Ingredients of proof II: G-arc-regular dihedrants with trivial cyclic core

Definition. For a group A and its subgroup $B \leq A$, the core of B in A is the largest normal subgroup of A contained in B, notation: $core_A(B)$. The subgroup $B \leq A$ is core-free if $\mathrm{core}_A(B) = 1$.

Theorem. (K, Marušič, and Muzychuk) Let $\Gamma = \mathrm{Cay}(D_{2n}, S)$ be a connected, G-arc-regular graph such that $(D_{2n})_* \leq G$, and $(C_n)_*$ is core-free in G. Then one of the following holds.

(i) $n = 1$, $\Gamma \cong K_2$, and $G \cong S_2$,

(ii) $n = 2$, $\Gamma \cong K_4$, and $G \cong A_4$,

(iii) $n = 3$, $\Gamma \cong K_{2,2,2}$, and $G \cong S_4$,

(iv) $n = 2m$, m is an odd number, $\Gamma \cong K_{n,n}$, and $G \cong (D_n \times D_n) \rtimes \langle \sigma \rangle$, where σ is an automorphism of $D_n \times D_n$ interchanging the coordinates (that is, $(x, y)^\sigma = (y, x)$ for all $(x, y) \in D_n \times D_n$).

Remark. By the above theorem we see that if $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ is a regular Cayley map such that n is odd, and $n > 3$, then $\mathrm{Aut}(\mathcal{M})$ has a non-trivial normal subgroup contained in $(C_n)_*$.

Ingredients of proof III: quotient maps

Definition. Let Γ be a graph, and $\mathcal{B}$ be a partition of $V(\Gamma)$. The quotient graph $\Gamma/\mathcal{B}$ is the graph (V, E) such that $V = \mathcal{B}$ and for two classes $B_1, B_2 \in \mathcal{B}$, $\{B_1, B_2\} \in E$ if and only if exists $\{v_1, v_2\} \in E(\Gamma)$ such that $v_1 \in B_1$ and $v_2 \in B_2$.

Ingredients of proof III: quotient maps

Definition. Let Γ be a graph, and $\mathcal{B}$ be a partition of $V(\Gamma)$. The quotient graph $\Gamma/\mathcal{B}$ is the graph (V, E) such that $V = \mathcal{B}$ and for two classes $B_1, B_2 \in \mathcal{B}$, $\{B_1, B_2\} \in E$ if and only if exists $\{v_1, v_2\} \in E(\Gamma)$ such that $v_1 \in B_1$ and $v_2 \in B_2$.

Definition. Let $\mathcal{M} = (\Gamma; R, L)$ be a regular map, and let $\mathcal{B}$ be a non-trivial, normal imprimitivity system of $\mathrm{Aut}(\mathcal{M})$. The relation $\sim_{\mathcal{B}}$ on $A(\Gamma)$ is defined by $(u_1, v_1) \sim_{\mathcal{B}} (u_2, v_2)$ if and only if exist block $B_1, B_2 \in \mathcal{B}$ such that $u_1, u_2 \in B_1$ and $v_1, v_2 \in B_2$.

Ingredients of proof III: quotient maps

Definition. Let Γ be a graph, and $\mathcal{B}$ be a partition of $V(\Gamma)$. The quotient graph $\Gamma/\mathcal{B}$ is the graph (V, E) such that $V = \mathcal{B}$ and for two classes $B_1, B_2 \in \mathcal{B}$, $\{B_1, B_2\} \in E$ if and only if exists $\{v_1, v_2\} \in E(\Gamma)$ such that $v_1 \in B_1$ and $v_2 \in B_2$.

Definition. Let $\mathcal{M} = (\Gamma; R, L)$ be a regular map, and let $\mathcal{B}$ be a non-trivial, normal imprimitivity system of $\mathrm{Aut}(\mathcal{M})$. The relation $\sim_{\mathcal{B}}$ on $A(\Gamma)$ is defined by $(u_1, v_1) \sim_{\mathcal{B}} (u_2, v_2)$ if and only if exist block $B_1, B_2 \in \mathcal{B}$ such that $u_1, u_2 \in B_1$ and $v_1, v_2 \in B_2$.

Proposition. Let Γ be a G-arc-regular graph, and $\mathcal{B}$ be a normal non-trivial imprimitivity system of G. If the stabilizer G_v is a Hamiltonian group for $v \in V(\Gamma)$, then the quotient graph $\Gamma/\mathcal{B}$ is $G^{\mathcal{B}}$-arc-regular.

Ingredients of proof III: quotient maps

Definition. Let Γ be a graph, and $\mathcal{B}$ be a partition of $V(\Gamma)$. The quotient graph $\Gamma/\mathcal{B}$ is the graph (V, E) such that $V = \mathcal{B}$ and for two classes $B_1, B_2 \in \mathcal{B}$, $\{B_1, B_2\} \in E$ if and only if exists $\{v_1, v_2\} \in E(\Gamma)$ such that $v_1 \in B_1$ and $v_2 \in B_2$.

Definition. Let $\mathcal{M} = (\Gamma; R, L)$ be a regular map, and let $\mathcal{B}$ be a non-trivial, normal imprimitivity system of $\mathrm{Aut}(\mathcal{M})$. The relation $\sim_{\mathcal{B}}$ on $A(\Gamma)$ is defined by $(u_1, v_1) \sim_{\mathcal{B}} (u_2, v_2)$ if and only if exist block $B_1, B_2 \in \mathcal{B}$ such that $u_1, u_2 \in B_1$ and $v_1, v_2 \in B_2$.

Proposition. Let Γ be a G-arc-regular graph, and $\mathcal{B}$ be a normal non-trivial imprimitivity system of G. If the stabilizer G_v is a Hamiltonian group for $v \in V(\Gamma)$, then the quotient graph $\Gamma/\mathcal{B}$ is $G^{\mathcal{B}}$-arc-regular.

Corollary. The relation $\sim_{\mathcal{B}}$ is R-invariant.

Definition. The quotient map $\Gamma/\mathcal{M}$ is the map

$$\mathcal{M}/\mathcal{B} = (\Gamma/\mathcal{B}; R^{\mathcal{B}}, L^{\mathcal{B}}),$$

where $R^{\mathcal{B}}$ is the permutation of $A(\Gamma/\mathcal{B})$ induced by R, and $L^{\mathcal{B}}$ is the dart-reversing involution switching the arcs of $\Gamma/\mathcal{B}$.

Definition. The quotient map $\Gamma/\mathcal{M}$ is the map

$$\mathcal{M}/\mathcal{B} = (\Gamma/\mathcal{B}; R^{\mathcal{B}}, L^{\mathcal{B}}),$$

where $R^{\mathcal{B}}$ is the permutation of $A(\Gamma/\mathcal{B})$ induced by R, and $L^{\mathcal{B}}$ is the dart-reversing involution switching the arcs of $\Gamma/\mathcal{B}$.

Corollary. Let $\mathcal{M} = (\Gamma; R, L)$ be a regular map, and $\mathcal{B}$ be a non-trivial, normal imprimitivity system of $\mathrm{Aut}(\mathcal{M})$. Then the quotient map $\mathcal{M}/\mathcal{B}$ is regular, and $\mathrm{Aut}(\mathcal{M}/\mathcal{B}) = \mathrm{Aut}(\mathcal{M})^{\mathcal{B}}$.

Definition. The quotient map $\Gamma/\mathcal{M}$ is the map

$$\mathcal{M}/\mathcal{B} = (\Gamma/\mathcal{B}; R^{\mathcal{B}}, L^{\mathcal{B}}),$$

where $R^{\mathcal{B}}$ is the permutation of $A(\Gamma/\mathcal{B})$ induced by R, and $L^{\mathcal{B}}$ is the dart-reversing involution switching the arcs of $\Gamma/\mathcal{B}$.

Corollary. Let $\mathcal{M} = (\Gamma; R, L)$ be a regular map, and $\mathcal{B}$ be a non-trivial, normal imprimitivity system of $\mathrm{Aut}(\mathcal{M})$. Then the quotient map $\mathcal{M}/\mathcal{B}$ is regular, and $\mathrm{Aut}(\mathcal{M}/\mathcal{B}) = \mathrm{Aut}(\mathcal{M})^{\mathcal{B}}$.

Corollary. Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be a regular Cayley map such that n is odd. Then $|S \cap C_n| = |S|/2$, and the set T of arc-inverting involutions has size $|T| = 2n$..

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

- G has an imprimitivity system $\mathcal{B}$ of block size 2.

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

- G has an imprimitivity system $\mathcal{B}$ of block size 2.
  By replacing $\mathcal{M}$ with a suitable isomorphic map, we may assume that $\mathcal{B} = \{1, r\}^G$.

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

- G has an imprimitivity system $\mathcal{B}$ of block size 2. By replacing $\mathcal{M}$ with a suitable isomorphic map, we may assume that $\mathcal{B} = \{1, r\}^G$.
- The kernel of G acting on $\mathcal{B}$ is $N = \langle \mu_{\mathrm{blue}}, \mu_{\mathrm{red}} \rangle$.

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

- G has an imprimitivity system $\mathcal{B}$ of block size 2.
  By replacing $\mathcal{M}$ with a suitable isomorphic map, we may assume that $\mathcal{B} = \{1, r\}^G$.
- The kernel of G acting on $\mathcal{B}$ is $N = \langle \mu_{\mathrm{b}lue}, \mu_{\mathrm{red}} \rangle$.
- The subgroup of G generated by the set T of arc-inverting involutions is $\langle s_*^G \rangle = N \rtimes D_*$.

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

- G has an imprimitivity system $\mathcal{B}$ of block size 2. By replacing $\mathcal{M}$ with a suitable isomorphic map, we may assume that $\mathcal{B} = \{1, r\}^G$.
- The kernel of G acting on $\mathcal{B}$ is $N = \langle \mu_{\mathrm{blue}}, \mu_{\mathrm{red}} \rangle$.
- The subgroup of G generated by the set T of arc-inverting involutions is $\langle s_*^G \rangle = N \rtimes D_*$.
- Let $G_1 = \langle \psi \rangle$. Then $\psi^4 = \sigma \in \mathrm{Aut}(D_{2n})$.

Proof of (i) of Theorem.

Let $\mathcal{M} = \mathrm{CM}(D_{2n}, S, p)$ be regular and not t-balanced for any t.

We reconstruct the group $G = \mathrm{Aut}(\mathcal{M})$:

- G has an imprimitivity system $\mathcal{B}$ of block size 2.
  By replacing $\mathcal{M}$ with a suitable isomorphic map, we may assume that $\mathcal{B} = \{1, r\}^G$.
- The kernel of G acting on $\mathcal{B}$ is $N = \langle \mu_{\mathrm{blue}}, \mu_{\mathrm{red}} \rangle$.
- The subgroup of G generated by the set T of arc-inverting involutions is $\langle s_*^G \rangle = N \rtimes D_*$.
- Let $G_1 = \langle \psi \rangle$. Then $\psi^4 = \sigma \in \mathrm{Aut}(D_{2n})$.
  $G = (N \rtimes D_*) \rtimes \langle \sigma \rangle$, and $\mathcal{M} \cong \mathrm{CM}(n, \ell)$. $\square$

Thank you!