# Strongly regular graphs with no triangles

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The Higman-Sims strongly regular graph $\Gamma$ with the parameters $(v, k, \lambda, \mu) = (100, 22, 0, 6)$ is well known since seminal paper from 1968. Less known is that the same graph $\Gamma$ was discovered by Dale Marsch Mesner (briefly DM) in his thesis of 1956.

While in 1956 the proof on the uniqueness of $\Gamma$ was almost reached by DM (three possible adjacency matrices were not inspected), the complete proof of uniqueness was given by DM in his mimeographed notes of year 1964.

Recently, together with Andy Woldar (and with support of Matan Ziv-Av) we investigated techniques and results of DM (1956, 1964), in particular those related to SRGs with no triangles (tfSRG), and found them still helpful and attractive.

We consider only primitive SRGs, that is those for which both graph and its complement are connected.

Standard parameters of SRG:

| | |
|---|---|
| $n$ | number of vertices, |
| $k$ | valency of vertex, |
| $\lambda$ | valency of edge, |
| $\mu$ | valency of non-edge. |

Fulfillment of classical integrability conditions is assumed.

We consider graph $\Gamma = (V, E)$, $x \in V$, $\Gamma(x)$ and $\Delta(x)$ are neighbours and non-neighbours of $x$.

Consider bipartite subgraph of $\Gamma$ with two parts $\Gamma(x)$ and $\Delta(x)$ as incidence graph of a suitable incidence structure $C$.

## Theorem 1 (2.6, DM, 1956)

(i) *C is a BIBD with the parameters*

$$\hat{v} = k$$
$$\hat{b} = l = n - k - 1$$
$$\hat{r} = k - \lambda - 1$$
$$\hat{k} = \mu$$
$$\hat{\lambda} = \mu - 1$$

*(caret ˆ denotes parameters of BIBD);*

(ii) *Any block of C is disjoint from at least $k - \mu$ blocks.*

9 parameter sets with $n \leq 100$:

| No. | $v$ | $k$ | $\lambda$ | $\mu$ | Existence |
|---|---|---|---|---|---|
| 1 | 5 | 2 | 0 | 1 | Yes, pentagon |
| 3 | 10 | 3 | 0 | 1 | Yes, Petersen |
| 16 | 16 | 5 | 0 | 2 | Yes, Clebsch |
| 15 | 28 | 9 | 0 | 4 | No, 1956 |
| 34 | 50 | 7 | 0 | 1 | ?, Hoffman-Singleton |
| 39 | 56 | 10 | 0 | 2 | ?, Gewirtz |
| 50 | 64 | 21 | 0 | 10 | No, 1956 |
| 64 | 77 | 16 | 0 | 4 | Yes, 1956 |
| 94 | 100 | 22 | 0 | 6 | Yes, 1956 |

Hussain's identities for BIBDs:

$$k(r-1)+k(k-1)(\lambda-1)=\sum_{l=0}^{k} l^2 f(l), \tag{2.41}$$

$$k(r-1)=\sum_{l=0}^{k} l f(l), \tag{2.42}$$

here $f(l)$ denotes the number of blocks of the design which have precisely $l$ common elements with a chosen block.

<u>Result</u> (p.61, 1956)
The graphs with the numbers 15 and 50 are impossible.

Proof is based on the variance counting, using results of Hussain.
**Remark** Modern way relies on Krein inequalities.

It was DM who suggested already in 1956 to consider SRGs of negative Latin square type with the parameters:

$$v = n^2$$

$$k = g(n+1)$$

$$\lambda = (g+1)(g+2) - n - 2$$

$$\mu = g(g+1)$$

Letting in addition $\lambda = 0$, DM gets

$$n = (g+1)(g+2) - 2 = g^2 + 3g = g(g+3)$$

Thus we obtain a one-parameter series of putative parameters for tfSRGs of negative Latin square type:

$$\begin{aligned}
n &= g(g+3),\\
v &= g^2(g+3)^2,\\
k &= g(g^2+3g+1),\\
\lambda &= 0,\\
\mu &= g(g+1).
\end{aligned}$$

The parameters of design $C$:

$$\hat{v} = g(g^2 + 3g + 1),$$

$$\hat{b} = (g^2 + 2g - 1)(g^2 + 3g - 1),$$

$$\hat{r} = (g + 1)(g^2 + 2g - 1),$$

$$\hat{k} = g(g + 1),$$

$$\hat{\lambda} = g^2 + g + 1.$$

Initial elements of theory of such tfSRGs appear already in 1956.
More transparent and general approach is presented in 1964.

**Proposition 2 (Equality 18.5, 1964)**

*Each block is disjoint from at least $g^2(g+2)$ other blocks.*

**Proposition 3 (Lemma 8.2, 1964)**

*Each block is disjoint from exactly $g^2(g+2)$ blocks and intersects each remaining block in exactly $g$ elements.*

(In modern terms: design $C$ is quasisymmetric)

## Theorem 4 (Theorem 8.3, 1964)

*The subgraph of non-neighbours of a vertex $x$ in $NL_g(g^2+3g)$ is a tfSRG with the parameters*
$\tilde{v} = (g^2+2g-1)(g^2+3g+1)$,
$\tilde{k} = g^2(g+2)$,
$\tilde{\lambda} = 0$,
$\tilde{\mu} = g^2$.

## Corollary 5 (Corollary 8.4.1, 1964)

*The existence of BIBD $C$ with the properties as above implies existence of $NL_g(g^2+3g)$.*

## Theorem 6 (Theorem 8.5, 1964)

*Existence of $NL_g(g^2+3g)$ implies the existence of 4 substructures which are BIBDs, namely:*

- *symmetric BIBD with $g^2(g+2)$ points,*
- *BIBD with $g(g+1)$ points and $(g+1)(g+2)(g^2+g-1)$ blocks,*
- *two BIBDs with $g^2(g+2)$ points and $(g+1)(g+2)(g^2+g-1)$ blocks.*

## Observation (1964)

Uniqueness of design $C$ with the requested properties implies uniqueness of $NL_g(g^2 + 3g)$.

**Remark.** Modern interpretation of results of DM includes the intersection diagram for the non-edge model of $NL_g(g^2 + 3g)$.

## Example 1

$$g = 1$$

$\hat{v} = 5$
$\hat{b} = 10$
$\hat{k} = 2$
$\hat{r} = 4$
$\hat{\lambda} = 1.$
This is trivial (complete) BIBD which is unique.

$$\Downarrow$$

Uniqueness of the Clebsch graph.

Motivation

SRGs with no triangles via DM

Negative Latin SRG series via DM

$NL_2(10)$: construction by DM

DM's construction in modern clothes

Subsequent results

Known tfSRGs and beyond

Graham Higman's ideas

Recent observations of Biggs

Miscellanea

In order to get graph $\Gamma = NL_2(10)$ with the parameters (100,22,0,6) it is necessary to start from a quasisymmetric design $C$ with 22 points and 77 blocks of size 6.

An example of such design was constructed by DM in 1956. He proved that up to isomorphism there are at most 4 such designs.

In 1964 DM proved the uniqueness of design $C$ with requested properties.

This implies uniqueness of graph $\Gamma$.

Discussion of construction by DM will be arranged below.

In fact, the design $C$, necessary to DM, was known for many years. This is Witt design $W_{22}$ (in fact 3-design), which was known also to Carmichael.

Witt proved (1938) uniqueness of $W_{22} = S(3, 6, 22)$ and that $Aut(W_{22}) = Aut(M_{22})$.

Knowledge of $W_{22}$ and its automorphism group was a starting point for Higman & Sims (1968) in the second (independent) appearance of $\Gamma$ and the group $Aut(\Gamma)$.

Recall that $Aut(\Gamma)$ contains new at that time sporadic simple group as a subgroup of index 2.

We describe graph $\Gamma$ as a merging of classes in a coherent configuration with 6 fibers of size 1, 1, 6, 16, 16, 60.

We keep the spirit of DM's presentation but not necessarily the language.

We use a computer: GAP (with GRAPE and nauty) and COCO.

Recall classical Sylvester's duads and synthemes:

- A duad is a subset of size 2 of a set of 6 elements.
- A syntheme is a partition of the set of six elements to 3 duads.
- In other words, an edge of $K_6$ and a 1-factor of $K_6$.
- Luckily, $\binom{6}{2} = (6-1)!! = 15$, so we have the same number of duads and synthemes.

An excursion to the properties of the number 6:

1-factorization of $K_6$

- 1-factorization of $K_6$ is a partition of its edge set into 5 1-factors.
- Simple combinatorics - there are exactly six 1-factorizations.
- They are all isomorphic.
- Clearly automorphism group of any 1-factorization has order 120, it is isomorphic to $S_5$.
- $Aut(S_6)$ interchanges two conjugacy classes of $S_5$ in $S_6$.

## Generalized quadrangle of order 2

- $GQ(2)$ has 15 points and 15 lines.
- Sylvester's model - 15 duads and 15 synthemes.
- $GQ(2)$ is unique up to isomorphism.
- Denoted by $W_2$.
- $GQ(2)$ is self-dual.
- $Aut(GQ(2)) \cong S_6$.
- Automorphism group of the incidence graph of $GQ(2)$ is $Aut(S_6)$.

## Spreads in $GQ(2)$

- A spread in an incidence structure is a set of blocks that form a partition of the point of sets.
- In Sylvester's model, a spread in $W_2$ is a 1-factorization of $K_6$.
- A spread in the dual, $W_2^T$ is a star.
- Stabilizer of a spread is $S_5$.

# Biplanes

- A biplane is a symmetric BIBD with $\lambda = 2$.
- In other words, an incidence structure such that:
  - Every block has $k$ points.
  - Every point is in $k$ blocks.
  - Every two points are in exactly two common blocks.
- A biplane with parameter $k$ has $\binom{k}{2} + 1$ points (and blocks).

# Biplanes on 16 points

- For $k = 6$, biplane has 16 points and 16 blocks.
- There are three such biplanes.
- All are self dual.
- Automorphism groups are of orders 384, 768 and 11520.
- The automorphism group of order $11520 = 16 \cdot 15 \cdot 48$ is 2-transitive.
- This is the highest symmetry such a biplane can have.
- Therefore we consider this biplane as the nicest biplane on 16 points.

# Nicest biplane on 16 points

- Clebsch graph $\square_5$ - Cayley graph $Cay(E_{16}, \{0001, 0010, 0100, 1000, 1111\})$.
- It is $NL_1(4)$.
- $Q_4$ plus diagonals.
- $Aut(\square_5) \cong E_{2^4} \rtimes S_5$.
- Clebsch graph is $SRG(16, 5, 0, 2)$.
- Our 16 points are 16 vertices.
- For each vertex, a block is the vertex and its 5 neighbours.
- Denote this incidence structure by $D = (\mathcal{B}, \mathcal{P})$.

Blocks of $D$ $\{0, 1, 2, 4, 8, 15\}$

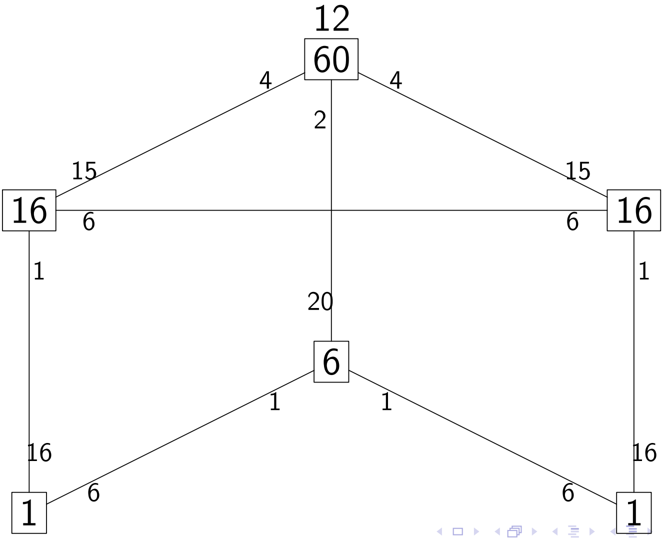
$\{0, 1, 3, 5, 9, 14\}$
$\{0, 2, 3, 6, 10, 13\}$
$\{1, 2, 3, 7, 11, 12\}$
$\{0, 4, 5, 6, 11, 12\}$
$\{1, 4, 5, 7, 10, 13\}$
$\{2, 4, 6, 7, 9, 14\}$
$\{3, 5, 6, 7, 8, 15\}$
$\{0, 7, 8, 9, 10, 12\}$
$\{1, 6, 8, 9, 11, 13\}$
$\{2, 5, 8, 10, 11, 14\}$
$\{3, 4, 9, 10, 11, 15\}$
$\{3, 4, 8, 12, 13, 14\}$
$\{2, 5, 9, 12, 13, 15\}$
$\{1, 6, 10, 12, 14, 15\}$
$\{0, 7, 11, 13, 14, 15\}$

# Ovals

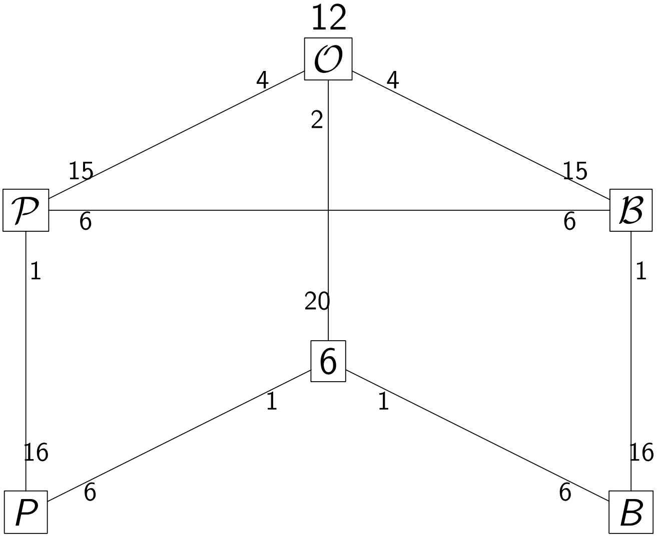
- An oval in a design is a set of points that intersect each block in exactly 2 or 0 points.
- In a biplane on 16 points an oval has 4 points.
- In $D$ ovals can be constructed by
  - taking an edge of $\square_5$: $\{a, b\}$.
  - Find 6 vertices not adjacent to both $a$ and $b$.
  - The induced subgraph on this set is a 1-factor.
  - Any edge of this one factor with $a$ and $b$ gives an oval.
- There are $\frac{16 \cdot 5 \cdot 3}{2 \cdot 2} = 60$ such ovals.
- Those are actually all ovals.
- Set of 60 ovals: $\mathcal{O}$.

Our aim is non-edge decomposition of $\Gamma = NL_2(10)$.

Intersection diagram:



# Actualization of the diagram (almost)



# Magical structure I

- We take 15 involutions of regular action of $E_{2^4}$.
- For each involution we take orbit - partition of point set to eight 2-sets.
- $\Sigma_1$ is set of such partitions.
- Consider incidence structure $(\mathcal{P}, \mathcal{O})$.
- Computation results:
  - There are 105 spreads in this structure. (Here spread is partition of $\mathcal{P}$ into 4 ovals of size 4).
  - The automorphism group of $(\mathcal{P}, \mathcal{O})$ has orbits of lengths 90 and 15.
  - The orbit of length 15 is a resolution of $\mathcal{O}$.

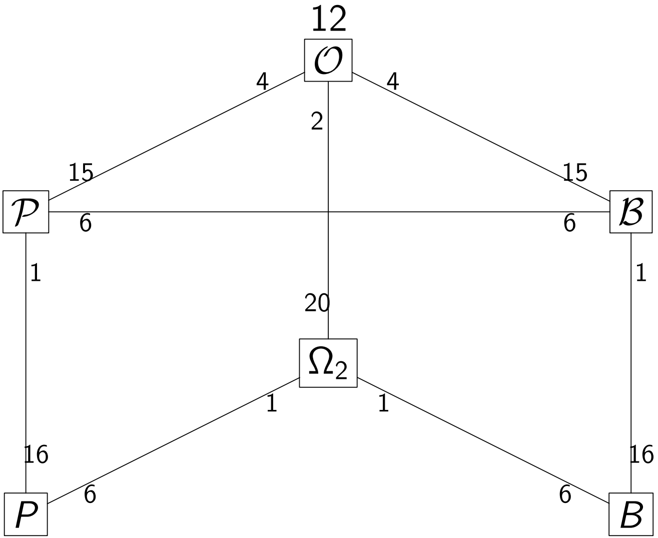
## Magical structure II

- We denote the orbit of length 15 by $\Sigma_2$.
- Incidence structure $(\Sigma_1, \Sigma_2)$ (with incidence defined by refinement), is $GQ(2)$.
- By computer search, we find a structure $M$ consisting of 5 elements of $\Sigma_2$: {{{0,1,10,11},{2,3,8,9},{4,5,14,15},{6,7,12,13}}, {{0,3,12,15},{1,2,13,14}, {4,7,8,11},{5,6,9,10}}, {{0,2,5,7}, {1,3,4,6},{8,10,13,15},{9,11,12,14}}, {{0,4,9,13},{1,5,8,12},{2,6,11,15}, {3,7,10,14}}, {{0,6,8,14},{1,7,9,15}, {2,4,10,12},{3,5,11,13}}}

## Magical structure III

- The orbit of $M$ under the group $G$ is of length 6.
- This orbit is denoted by $\Omega_2$.
- Every element of $\Sigma_2$ is in exactly two elements of $\Omega_2$, therefore
- Every element of $\mathcal{O}$ is in exactly two elements of $\Omega_2$.

# Actualization of the diagram



## Adjacency in the graph

- A vertex from $\mathcal{P}$ is adjacent to 15 ovals that it is incident to.
- A vertex from $\mathcal{B}$ is adjacent to 15 ovals to which it is disjoint.
- Two ovals are adjacent provided they are disjoint and do not occur in any common magical structure $\in \Omega_2$.
- Thus, regular graph of valency 22.
- No triangles.
- $\mu = 6$.

DM was not aware that his design $C$ is (a quasisymmetric) Steiner system.

## Lemma 7

*For a 2-design $D$, any two of the following imply the third:*

(i) *$D$ is a 3-design;*

(ii) *$D$ is a QSD with $x = 0$;*

(iii) *$D$ has $\frac{v(v-1)}{k}$ blocks.*

Proof is known for a few decades, see e.g. Cameron, van Lint (1991).

Oppositely to DM, a common way to construct $W_{22}$ goes via procedure of a transitive extension. This is a way exploited by Witt.

First steps in deeper understanding of the extension construction were done by D. R. Hughes and H. Luneburg, relying on a model of W. L. Edge for $PG(2,4)$.

It was P. Cameron who considered systematically extension of symmetric designs.

An infinite series, requested by DM, is nothing else but his main one-parameter infinite series of feasible parameters of extensible symmetric BIBDs with $v = (\lambda + 2)(\lambda^2 + 4\lambda + 2)$ points.

Cameron also developed elements of a general theory of tfSRGs, a short outline appears in the book with van Lint (1991).

This is a very significant source of information. Nevertheless, many facts about such graphs, which go back to DM were for decades remaining in shadow, not vividly visible to researchers.

In our eyes, main input of DM is the consideration of a “combinatorial amalgam” instead of a group theoretical amalgam.

$$
\begin{tikzcd}
 & G = Aut(\Gamma) \arrow[dl, dash] \arrow[dr, dash] & \\
G_a = Aut(W_{22}) \arrow[dr, dash] & & G_{\{a,b\}} = E_{32} \ltimes S_6 \arrow[dl, dash] \\
 & G_{a,b} = Aut(D) = E_{16} \ltimes S_6 &
\end{tikzcd}
$$

In modern terms, DM constructed coherent configuration with 6 fibers, determined by $G_{a,b}$.

He found two subconfigurations, determined by $G_a$ and $G_{\{a,b\}}$. The intersection of these subconfigurations leads strictly to $\Gamma = NL_2(10)$ (use of coherent algebras provides a way for more accurate wording.)

No groups appear in DM's presentation, though color matrices are strictly visible.

Moreover, the suggested representation of DM's model provides a combinatorial substitution of the natural homomorphism from the group $Aut(D)$ onto $S_6$.

They all are induced subgraphs of $\Gamma = NL_2(10)$.

- Graphs on 56 and 77 vertices appear via edge and non-edge decompositions respectively.
- $\Gamma$ can be "divided" into two copies of Hoffman-Singleton graphs on 50 vertices.
- Pentagon and Petersen graphs are subgraphs of Clebsch graph.
- Extraction of Clebsch graph is more tricky and requires a special attention.

Construction of a new tfSRG remains a great challenge in modern algebraic graph theory.

First, what is coming to mind:

to go ahead with $NL_g(g^2 + 3g)$.

## Case $g = 3$

- An attractive lead for a few decades was to start from a biplane on 56 points and to extend it to a 3-design on 57 points.
- 5 such biplanes were known.
- B. Bagchi twice claimed that they are not extensible (see also Cameron & van Lint, 1991).
- A fatal flaw was detected by A. E. Brouwer.

Finally, P. Kaski and P. Ostergard proved that there exist exactly 5 biplanes on 56 points, no one of them is extensible.

(This is done via exhaustive computer search: 316 machines running in parallel for two months.)

Thus $NL_3(18)$ does not exist!

Next open case in DM's series is $g = 4$.

We wish $NL_4(28)$.

One who wishes to imitate DM's way of non-edge decomposition has to start from symmetric BIBD on 96 points (instead of the nicest biplane on 16 points).

Millions of such designs are available via the approach coined by W. Wallis, late D. Fon-Der-Flaass and extended by M. Muzychuk.

At the beginning highly symmetrical examples of such designs may be inspected.

The original result of G. Higman was unpublished, it appears in book of Cameron (1999).

The goal is to provide necessary conditions for the existence of certain automorphisms of putative SRG with given parameters.

Structure of subgraphs, induced by fixed points, is of a great significance.

Original result of Higman was a proof of the fact that there is no vertex-transitive tfSRG of valency 57 (Moore graph of valency 57).

These techniques are nowadays exploited by followers of Higman, namely by A. Makhnev et al and recently by M. Mačaj and J. Širáň.

Striking restrictions for the order of the automorphism group of a putative Moore graph of valency 57 are obtained by Mačaj and Širáň.

It is a good time to start hunting for such a graph relying on their results.

Recently Biggs provided a new input to theory of tfSRGs.

In particular, he gave direct proof of Krein conditions for such graphs and characterized those of them, which are Smith graphs (or 3-isoregular) that is second constituent is also an SRG.

Biggs also reconsidered question of listing of feasible parameters for such graphs. (He is not counting pentagon).

There are 21 parameter sets with at most 1000 vertices and a few dozens with fewer than 6025 vertices.

The smallest open case is a tfSRG with
$v = 162$
$k = 21$
$\lambda = 0$
$\mu = 3.$

Mačaj (extending Makhnev et al) obtained very strong restrictions on the order and structure of automorphisms of such putative graph.

A number of other problems related to tfSRGs look also very attractive.
Below we simply mention just a few of them.

- Use of spectral techniques, in particular eigenspaces of the second subconstituents. Pioneering input was provided by A. D. Gardiner.
- Investigation of outindependent subgraphs of SRGs by M. A. Fiol and E. Garriga with a striking application to a putative Moore graph of valency 57.
- Splitting of a tfSRG into two copies of tfSRGs (generalization of $100 = 50 + 50$). Here pioneering input belongs to R. Noda (1984).

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