A characterization of Leonard pairs using the notion of a tail

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Introduction

This talk is about Leonard pairs and Leonard systems, which are defined using linear algebra.

Leonard pairs have applications to such areas as distance-regular graphs, orthogonal polynomials, and representation theory.

In this talk, we characterize the Leonard pairs using the notion of a tail. This notion was originally introduced in the context of distance-regular graphs.

The main theorem can be viewed as an algebraic version of a theorem of Jurišić, Terwilliger, and Zitnik proven for Q-polynomial distance-regular graphs in 2009.

Preliminaries

We begin by establishing some preliminaries.

- Let $\mathbb{K}$ denote a field.
- Fix a vector space V of dimension $d+1$ $(d \geq 1)$.
- Let $\text{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra of all $d+1$ by $d+1$ matrices with entries in $\mathbb{K}$.
- Let $\mathcal{A} = \text{End}(V)$. Note that $\mathcal{A}$ is isomorphic to $\text{Mat}_{d+1}(\mathbb{K})$.

Tridiagonal matrices

- A square matrix is **tridiagonal** whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal.
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Example

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & -7 & 4 & 2 \\ 0 & 0 & 6 & -9 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 8 & 6 & 5 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

The matrix on the left is irreducible tridiagonal, but the one on the right is not.

Definition of a Leonard pair

Definition

Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a **Leonard pair** on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following properties:

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
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Note

It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are not using this convention.

Idempotents

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Definition

By a **system of mutually orthogonal idempotents** in $\mathcal{A}$, we mean a sequence $\{E_i\}_{i=0}^{d}$ of elements in $\mathcal{A}$ such that

$$E_i E_j = \delta_{i,j} E_i \qquad (0 \leq i, j \leq d),$$

$$rank(E_i) = 1 \qquad (0 \leq i \leq d).$$

Decompositions

Definition

By a **decomposition of V**, we mean a sequence $\{U_i\}_{i=0}^{d}$ consisting of one-dimensional subspaces of V such that

$$V = \sum_{i=0}^{d} U_i \qquad \text{(direct sum).}$$

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Given a decomposition, the projections onto the components are mutually orthogonal idempotents. Conversely, a decomposition can be defined from a set of mutually orthogonal idempotents.

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Definition

Let A denote a multiplicity-free element of $\mathcal{A}$ and let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A. For $0 \leq i \leq d$, let U_i denote the eigenspace of A for θ_i. Then $\{U_i\}_{i=0}^d$ is a decomposition of V; let $\{E_i\}_{i=0}^d$ denote the corresponding system of idempotents. We refer to E_i as the **primitive idempotent** of A corresponding to U_i (or θ_i).

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Note

$A = \sum_{i=0}^{d} \theta_i E_i$ and $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq d$.

Leonard systems

Definition

By a **Leonard system** on V, we mean a sequence

$$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

which satisfies the following properties.

1. Each of A, A^* is a multiplicity-free element of $\mathcal{A}$.
2. $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.
3. $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.
4. $$E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i,j \leq d).$$
5. $$E_i A^* E_j = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i,j \leq d).$$

Leonard systems (continued)

Note

A Leonard system gives rise to a Leonard pair and vice versa. Also, if A, A^* are a Leonard pair, then A and A^* are multiplicity-free.

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$$E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1; \\ \neq 0, & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i, j \leq d).$$

Let $\{\theta_i^*\}_{i=0}^d$ denote scalars in $\mathbb{K}$ and let $A^* = \sum_{i=0}^d \theta_i^* E_i^*$. To avoid trivialities, assume that $d \geq 1$.

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Facts

- E_0^*, A generate $\mathcal{A}$.
- There exists an antiautomorphism $\dagger$ of $\mathcal{A}$ that fixes E_i, A^*.

The graph Δ

We will now examine a set of conditions necessary to guarantee that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system. To do so, we first define a graph Δ.

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Note

The graph Δ is finite and undirected, without loops or multiple edges. Also, Δ is well-defined because of the antiautomorphism $\dagger$.

Leonard systems and Δ

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Lemma

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The sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system if and only if the graph Δ is a path such that vertices $i-1, i$ are adjacent for $1 \leq i \leq d$.

The "only if" statement and most of the details of the "if" statement follow from the definitions.

The bulk of the work in proving the "if" statement involves proving that A^* is multiplicity-free. This is accomplished by showing that the minimal polynomial for A^* has distinct roots and is degree $d+1$.

Q-polynomial properties

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The given ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents of A is said to be **Q-polynomial** whenever the equivalent conditions in the previous lemma hold.

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Definition

Let (E, F) denote an ordered pair of distinct primitive idempotents for A. This pair will be called **Q-polynomial** whenever there exists a Q-polynomial ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents of A such that $E = E_0$ and $F = E_1$.

Tails

Definition

Let $(E, F) = (E_i, E_j)$ denote an ordered pair of distinct primitive idempotents for A. This pair will be called a **tail** whenever the following occurs in Δ:

1. i is adjacent to no vertex in Δ besides j;
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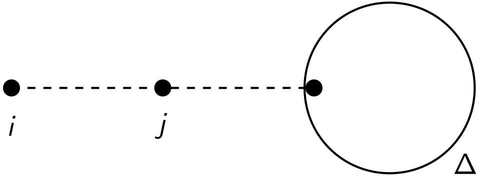

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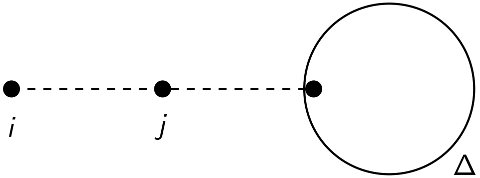

Lemma

Let (E, F) denote an ordered pair of distinct primitive idempotents for A. If (E, F) is Q-polynomial, then (E, F) is a tail.

The main theorem

Theorem

Let (E, F) denote an ordered pair of distinct primitive idempotents for A. Then this pair is Q-polynomial if and only if the following hold.

1. *(E, F) is a tail.*
2. *There exists $\beta \in \mathbb{K}$ such that $\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ is independent of i for $1 \leq i \leq d-1$.*
3. *$\theta^*_0 \neq \theta^*_i$ for $1 \leq i \leq d$.*

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3. *$\theta^*_0 \neq \theta^*_i$ for $1 \leq i \leq d$.*

The "only if" statement follows mainly from the definitions, except for condition 2, which was shown by Terwilliger (2001).

The "if" statement can be proven by showing that conditions 1–3 imply that Δ is a path.

It is first necessary to show that Δ is connected.

Proof outline part 2

Each connected component of Δ corresponds to a subspace of V that is invariant under both A and A^*. So, if Δ is not connected then there exists a subspace U of V such that $U \neq 0$, $U \neq V$, $AU \subseteq U$, and $A^*U \subseteq U$.

An important consequence of condition 3 is that both A and A^* generate $\mathcal{A}$. Therefore, $\mathcal{A}U \subseteq U$. Because V is irreducible as an $\mathcal{A}$-module, either $U = 0$ or $U = V$. This is a contradiction, so Δ is connected.

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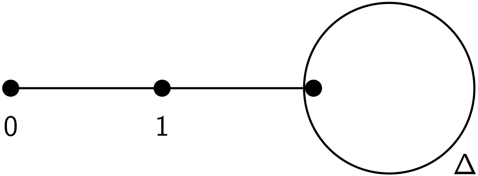

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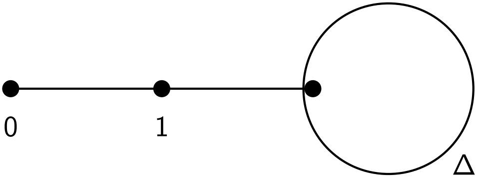

To show that Δ is a path, first let

$$\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} \qquad (1 \le i \le d-1)$$

Proof outline part 4

Using the three-term recurrence (condition 2), it can be shown that the expression

$$\theta_{i-1}^{*2} - \beta\theta_{i-1}^{*}\theta_i^{*} + \theta_i^{*2} - \gamma^{*}(\theta_{i-1}^{*} + \theta_i^{*})$$

is independent of i for $1 \leq i \leq d$. Call this expression δ^{*}.

Proof outline part 4

Using the three-term recurrence (condition 2), it can be shown that the expression

$$\theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i)$$

is independent of i for $1 \le i \le d$. Call this expression δ^*.

Next, it follows that

$$0 = [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^*A],$$

where $[x, y] = xy - yx$.

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where $[x, y] = xy - yx$.

This can be proven using the following facts:

- $I = \sum_{i=0}^{d} E^*_i$,
- $E^*_i A^* = \theta^*_i E^*_i$, and
- $A^* E^*_j = \theta^*_j E^*_j$.

Proof outline part 5

Suppose we are given vertices i and j in Δ that are path-length distance 3 and there exists a unique path (i, r, s, j) of length 3 connecting i and j. Then

$$\theta_i - (\beta + 1)\theta_r + (\beta + 1)\theta_s - \theta_j = 0.$$

Proof outline part 5

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$$\theta_i - (\beta + 1)\theta_r + (\beta + 1)\theta_s - \theta_j = 0.$$

To show this, expand the right-hand side of

$$0 = [A^*, A^{*2}A - \beta A^* A A^* + A A^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A]$$

and left and right multiply by E_i and E_j, respectively, and simplify.

Proof outline part 6

We can now easily show that Δ is a path by showing that every vertex in Δ is adjacent to at most two other vertices. Suppose there exists a vertex i in Δ that is adjacent to at least three other vertices. Without loss of generality, assume that this setup looks like this:

Proof outline part 6

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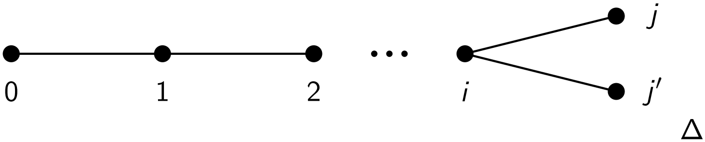

Proof outline part 7

Given the previously-established algebraic facts,

$$\theta_{i-2} - (\beta + 1)\theta_{i-1} + (\beta + 1)\theta_i - \theta_j = 0$$

and

$$\theta_{i-2} - (\beta + 1)\theta_{i-1} + (\beta + 1)\theta_i - \theta_{j'} = 0.$$

Comparing these equations, we find $\theta_j = \theta_{j'}$. Recall that $\{\theta_h\}_{h=0}^{d}$ are mutually distinct, so $j = j'$. This is a contradiction and we have now shown that Δ is a path.

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The ordering of primitive idempotents $E_0, E_1, \ldots$ induced by the path is Q-polynomial, so the pair $(E, F) = (E_0, E_1)$ is Q-polynomial.

Summary

We defined the closely related notions of Leonard pairs and Leonard systems. We then relaxed the definition of a Leonard system by considering a system based on two linear transformations, but which only satisfies half of the properties.

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