Locally distance transitive graphs

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s-arc transitive graphs

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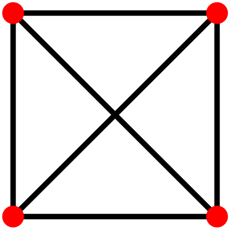

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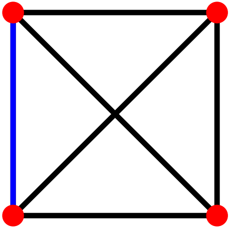

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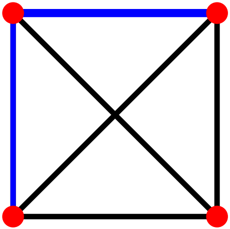

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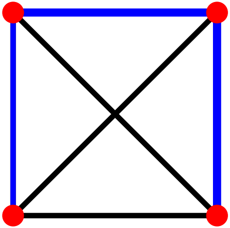

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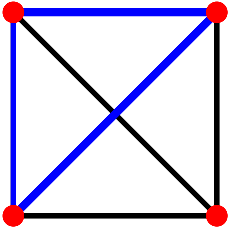

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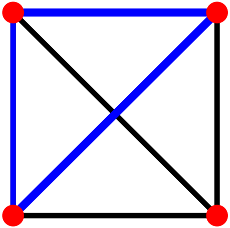
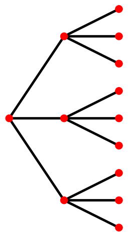


K_4 is 2-arc transitive but not 3-arc transitive.

s-distance transitive graphs

Γ is called s-distance transitive if for each $i \leq s$, $\mathrm{Aut}(\Gamma)$ is transitive on the set $\{(v, w) \mid d(v, w) = i\}$.



s-arc transitive vs s-distance transitive

If $s \leq \lfloor \frac{g-1}{2} \rfloor$, where g is the girth, then Γ is s-distance transitive if and only if Γ is s-arc transitive.

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The point graph of the $G_2(q)$ hexagon is 3-distance transitive but not 2-arc transitive.

Distance transitive graphs

If Γ is *s*-distance transitive for all $s \leq \mathrm{diam}(\Gamma)$ then Γ is distance transitive.

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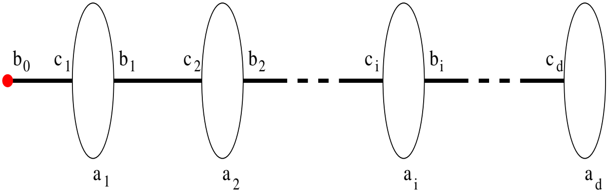

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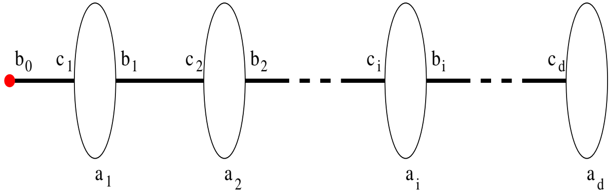

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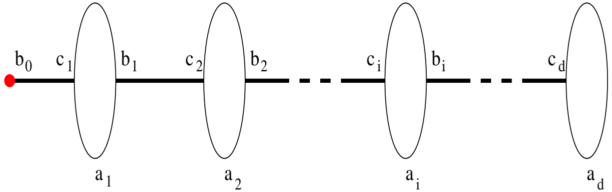

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For s-distance transitive graphs the parameters are only well defined out to distance s.

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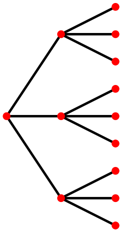

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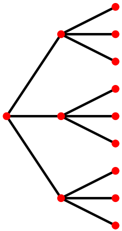

If $s \leq \lfloor \frac{g-1}{2} \rfloor$, where g is the girth, then Γ is locally s-distance transitive if and only if Γ is locally s-arc transitive.

Local symmetry II

A locally s-distance transitive connected graph is edge-transitive and so either

- Γ is vertex-transitive and s-distance-transitive
- $\mathrm{Aut}(\Gamma)$ has two orbits on vertices and Γ is bipartite.

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For $G \leqslant \mathrm{Aut}(\Gamma)$ we can also refer to locally (G, s)-distance transitive.

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Vertex-intransitive case

The distance parameters for a vertex only depend on the part of the bipartition it belongs to.

eg line-plane incidence graph of a projective space.

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Shawe-Taylor (1987)

- The distance 2 graph has two connected components, each of which is distance transitive.
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Uses work of Smith, Praeger-Saxl-Yokoyama on distance transitive graphs.

Quotients

Let $\mathcal{B}$ be a partition of $V\Gamma$

Quotient graph $\Gamma_{\mathcal{B}}$:

vertex set: parts of $\mathcal{B}$

adjacency: $B_1 \sim B_2$ if there exists $v_1 \in B_1$ and $v_2 \in B_2$ such that $v_1 \sim v_2$.

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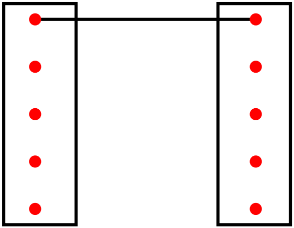

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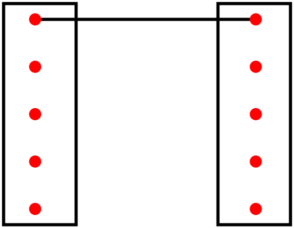

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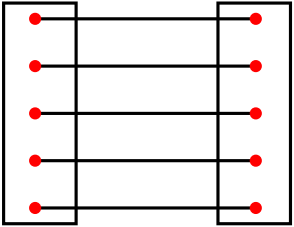

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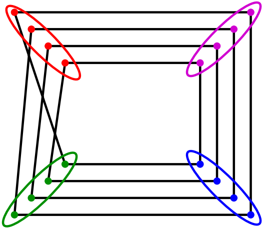
For each family, can reduce to a graph in the family that has no meaningful quotients (basic) and then study the basic graphs in the family.

This usually involves knowledge of quasiprimitive groups.

Quotients of locally s-distance transitive graphs?

Paths in Γ may decrease in length in Γ_N and indeed Γ_N may have smaller diameter than Γ.



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Let $LDT(s)$ be the set of graphs Γ that are locally s'-distance transitive where $s' = \min\{s, \mathrm{diam}(\Gamma)\}$.

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Theorem (Devillers-Giudici-Li-Praeger)

Let $s \geq 2$ and let $\Gamma \in LDT(s)$ relative to G and let $N \lhd G$ with at least three orbits on vertices. Then one of the following holds:

- $\Gamma = K_{m[b]}$,
- Γ_N *is a star,*
- $\Gamma_N \in LDT(s)$ *relative to* G/N *and* Γ *is a cover of* Γ_N.

Basic and degenerate graphs

By a degenerate graph we mean K_1, K_2 or a star $K_{1,m}$.

Γ is a basic locally (G, s)-distance transitive graph if each only normal quotient is one of these degenerate graphs.

Basic graphs

There are four types of basic locally (G, s)-distance transitive graphs to study:

- G acts quasiprimitively on $V\Gamma$;
- Γ is bipartite, G is biquasiprimitive on $V\Gamma$ and G^+ acts faithfully on each orbit;
- Γ is bipartite, $G = G^+$ acts faithfully and quasiprimitively on each orbit;
- Γ is bipartite, $G = G^+$ acts faithfully on both orbits and quasiprimitively on only one.

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These are currently under investigation.

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G^+ may or may not act quasiprimitively on each orbit.

If G^+ is not quasiprimitive on each orbit then can take normal quotients with respect to G^+

ie, Γ is G-basic but not G^+-basic.

Coset graphs

- G a group with subgroup H,
- $g \in G \backslash H$ such that $g^2 \in H$.

We can construct the graph $Cos(G, H, HgH)$ with

vertex set: cosets of H in G
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Any arc-transitive graph Γ can be constructed in this way:

- $G = \mathrm{Aut}(\Gamma)$, $H = G_v$
- g an element interchanging v and w, where $\{v, w\} \in E\Gamma$.

An example

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Now choose $u \neq b$ in the Sylow 2-subgroup of T containing b.

Let $g = (u, ub)\sigma \in G$.

Define $\Gamma = Cos(G, L, LgL)$.

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Γ is a cover of a locally $(T, 2)$-arc transitive graph.