Semiregular Actions in Combinatorial Structures

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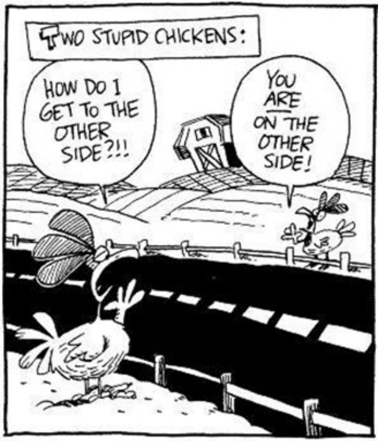
What is symmetry?


TWO STUPID CHICKENS:
HOW DO I GET TO THE OTHER SIDE?!!
YOU ARE ON THE OTHER SIDE!

Contents

Regular Actions

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Definition

An action of a group on a set is said to be regular if it is transitive and the stabilizers of all the elements of the set are trivial.

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Definition

More generally, an action is semiregular if the stabilizers of all the elements of the set are trivial.

Contents

2-designs

2-designs

Definition

A (v, k, λ) *design on a set of points* X *of cardinal* v *is a family* $B_1, \ldots, B_b$ *of* k*-subsets of* X*, called blocks, such that any two distinct points of* X *are in exactly* λ *blocks.*

2-designs

Definition

A (v, k, λ) design on a set of points X of cardinal v is a family $B_1, \ldots, B_b$ of k-subsets of X, called blocks, such that any two distinct points of X are in exactly λ blocks.

Definition

If G is a group of order v, a (v, k, λ) difference family on G is a family $B_1, \ldots, B_s$ of k-subsets of G (called base blocks) such that any non-identity element of G can be expressed in exactly λ ways as a difference of elements of some base block.

2-designs

Definition

If $B_1, \ldots, B_s$ is a (v, k, λ)-difference family on G, its development is $\{B_i + g \mid i = 1, \ldots, s, g \in G\}$

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Proposition

The development of a (v, k, λ) difference family on G is a (v, k, λ)-design on G.

2-designs

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Proposition

The development of a (v, k, λ) difference family on G is a (v, k, λ)-design on G. This group G acts regularly on the points of the design and, if there is not repetitions in the blocks, it acts also semiregularly on the blocks.

2-designs

Proposition

If (X, B) is a 2-design admitting G as a group of automorphisms acting regularly on the points and semiregularly on the blocks, then the design is isomorphic to the development of certain difference family on G.

2-designs

Proposition

(Martínez, Djokovic, Vera-López) If $q = 3^n + 2$ is a prime power with n odd and $q' = 3^{n+1}$ then, when we consider the cyclotomy of order 4 in $\mathbb{F}_q \times F_{q'}$, we have that

$$\{B_1 = \mathbb{F}_q^\star \cup C_0, B_2 = \mathbb{F}_q^\star \cup C_3\}$$

is a $(qq', \frac{qq'+3}{4}, \frac{qq'+3}{8})$-difference family in the additive group of $\mathbb{F}_q \times F_{q'}$.

2-designs

Definition

If $\{B_1, \ldots, B_s\}$ *is a* (v, k, λ)*-difference family on an abelian group* G*, an integer* t *prime to* v *is said to be a multiplier if for every* i *in* $\{1, \ldots, s\}$ *there exists* j *in* $\{1, \ldots, s\}$ *and there exists* g_i *in* G *such that* $tB_i = B_j + g_i$.

2-designs

Definition

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2-designs

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Proposition

(Martínez, Djokovic, Vera-López) If $\{B_1, \ldots, B_s\}$ *is a* (v, k, λ)*-difference family on a group* G*,* U *is a normal subgroup of* G *of odd index* u *and* t *is a cyclic multiplier that fixes the cosets of* U *in* G*, then there exists integers* x, y, z *not all zero such that*

$$sx^2 = (sk - \lambda)y^2 + (-1)^{\frac{n-1}{2}}\lambda uz^2.$$

Contents

graphs

Definition

An m-Cayley graph respect to a group G is a graph (V, E) admitting a group of automorphisms isomorphic to G that acts semiregularly on the set of points of the graph.

graphs

if $p_1, \ldots, p_m$ are representatives of the respective orbits, then the graph can be determined by the set $\{S_{i,j}\}$ of subsets of G defined by

$$S_{i,j} = \{g \in G \mid p_i^g p_j \in E\}.$$

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This set is called the symbol of the graph.
When $m = 2$ we will denote the symbol by $[Q, R, S]$, where $Q = S_{1,0}, R = S_{0,0}$ and $S = S_{1,1}$.

graphs

Definition

A graph (V, E) *is a* (v, k, λ, μ)*-strongly regular graph if* $|V| = v$*, is* k*-regular and if, for any distinct* x *and* y *in* V, $|\{z \in V \mid xz, zy \in E\}|$ *is* λ *if* $xy \in E$ *and* μ *in another case. In this situation, we will set* $\beta = \lambda - \mu, \gamma = k - \mu$ *and* $\Delta = \sqrt{b^2 + 4\gamma}$.

graphs

Definition

$[Q, R, S]$ is a partial difference triple if it satisfies

$$\begin{aligned} R^2 - Q^2 &= \gamma e + \beta R + \mu G && (1) \\ Q(R+S) &= \beta Q + \mu G && (2) \\ S^2 - Q^2 &= \gamma e + \beta S + \mu G. && (3) \end{aligned}$$

in the group ring $\mathbb{Z}[G]$ for certain integers γ, β and μ.

graphs

Definition

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in the group ring $\mathbb{Z}[G]$ for certain integers γ, β and μ.

Proposition

$[Q, R, S]$ is a partial difference triple iff it is the symbol of a (v, k, λ, μ)-strongly regular graph.

graphs

Proposition

(Leung and Ma) Up to complementation, the parameters for any nontrivial partial difference triples in cyclic groups are the following:

1. $(n; r, s; \lambda, \mu) = (2m^2+2m+1; m^2, m^2+m; m^2-1, m^2), m \geq 1.$
2. $(n; r, s; \lambda, \mu) = (2m^2; m^2, m^2-m; m^2-m, m^2-m), m \geq 2.$
3. $(n; r, s; \lambda, \mu) = (2m^2; m^2, m^2+m; m^2+m, m^2+m), m \geq 3.$
4. $(n; r, s; \lambda, \mu) = (2m^2; m^2 \pm m, m^2; m^2 \pm m, m^2 \pm m), m \geq 2.$

, where $n = |G|, r = |R|$ and $s = |S|$.

graphs

If $m = 3$ we will denote the symbol by $[A, B, C; R, S, T]$, where $A = S_{0,0}, B = S_{1,1}, C = S_{2,2}, r = S_{1,0}, S = S_{2,1}, T = S_{0,2}$, and $TCay(\mathbb{Z}_n; A, B, C; R, S, T)$ will denote the tri-Cayley graph defined by that symbol.

graphs

Proposition

(K. Kutnar, D. Marusic,S. Miklavic, P. Sparl) $[A, B, C; R, S, T]$ *is the symbol of a* $(3n, k, \lambda, \mu)$*-strongly regular graph iff the following identities hold in the group ring* $\mathbb{Z}[G]$*:*

1. $A^2 + R^{(-1)}R + TT^{(-1)} = \beta A + \mu G + (k - \mu)e$
2. $B^2 + S^{(-1)}S + RR^{(-1)} = \beta B + \mu G + (k - \mu)e$
3. $C^2 + T^{(-1)}T + SS^{(-1)} = \beta C + \mu G + (k - \mu)e$
4. $RA + BR + S^{(-1)}T^{(-1)} = \beta R + \mu G$
5. $SB + CS + T^{(-1)}R^{(-1)} = \beta S + \mu G$
6. $TC + AT + R^{(-1)}S^{(-1)} = \beta T + \mu G$

graphs

Proposition

(K. Kutnar, D. Marusic,S. Miklavic, P. Sparl) Let $X = TCay(\mathbb{Z}_n; A, B, C; R, S, T)$ *be a nontrivial* $(3n, k, \lambda, \mu)$*-strongly regular tricirculant, where* $|A| + |B| + |C| \leq |\overline{A}| + |\overline{B}| + |\overline{C}|$. *Assume* $A \cup B \cup C \neq \emptyset$. *If n is a prime or n is coprime to* 6Δ *then there exists an integer s such that the following two statements hold:*

graphs

Proposition

1. *If the cardinalities of* A, B *and* C *are not all equal, then* $(3n, k, \lambda, \mu) = (3(12s^2+9s+2), (4s+1)(3s+1), s(4s+3), s(4s+1))$. *If in addition* $|A| = |B| \neq |C|$ *(which is equivalent to* $|R| = |S| \neq |T|$*), then* $|A| = 2s(1+2s), |B| = (4s+1)(3s+1), |R| = s(4s+1)$ *and* $|T| = (1+2s)^2$.
2. *If* $|A| = |B| = |C|$ *then* $(3n, k, \lambda, \mu) = (3(3s^2-3s+1), s(3s-1), s^2+s-1, s^2)$. *In this case* $|A| = s(s-1)$ *and* $|R| = |S| = |T| = s^2$.

Contents

digraphs

Definition

An m-Cayley digraph respect to a group G is a digraph (V, E) admitting a group of automorphisms isomorphic to G that acts semiregularly on the set of points of the digraph.

digraphs

Conjecture (The semiregularity problem)

Every vertex transitive digraph has a semiregular automorphism.

digraphs

if $p_1, \ldots, p_m$ are representatives of the respective orbits, then the digraph can be determined by the set $\{S_{i,j}\}$ of subsets of G defined by

$$S_{i,j} = \{g \in G \mid p_i^g p_j \in E\}.$$

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This set is called the symbol of the digraph.

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$$S_{i,j} = \{g \in G \mid p_i^g p_j \in E\}.$$

This set is called the symbol of the digraph.
When $m = 2$ we will denote the symbol by $[Q, R, S, T]$, where $Q = S_{1,0}, R = S_{0,0}, S = S_{1,1}$ and $T = S_{0,1}$.

graphs

Definition

(Duval) A digraph (V, E) is a (v, k, μ, λ, t)-strongly regular digraph if $|V| = v$, is k-regular and if, for any x in V, $|\{z \in V \mid xz, zx \in E\}|$ is t and for any distinct x and y in V, $|\{z \in V \mid xz, zy \in E\}|$ is λ if $xy \in E$ and μ in another case. In this situation, we will set $\beta = \lambda - \mu$ and $\gamma = t - \mu$.

digraphs

Definition

(L. Martínez, A. araluze) Let H be a group of order n and m an integer with $m \geq 1$. A family $\mathfrak{S} = \{S_{i,j}\}$, with $0 \leq i, j < m$, of subsets of H is a $(m, n, k, \mu, \lambda, t)$-partial sum family (for short, $(m, n, k, \mu, \lambda, t)$-PSF, or simply PSF if we do not specify the parameters) if it satisfies:

1. *for every i it holds that $e \notin S_{i,i}$, where e is the identity of H.*
2. *for every i it holds that $\sum_{j=0}^{m-1} |S_{i,j}| = \sum_{j=0}^{m-1} |S_{j,i}| = k$*
3. *for every i and j it holds that $\sum_{k=0}^{m-1} S_{i,k} S_{k,j} = \delta_{i,j} \gamma e + \beta S_{i,j} + \mu H$, where $\delta_{i,j}$ is the Kronecker delta and e is the identity of the group.*

When $m = 2$ they are called Partial Sum Quadruples (PSQs).

digraphs

Proposition

$\mathfrak{S} = \{S_{i,j}\}$ is a $(m, n, k, \mu, \lambda, t)$-partial sum family iff it is the symbol of a (v, k, μ, λ, t)-strongly regular digraph.

digraphs

Proposition

(A. Araluze, K. Kutnar, L. Martínez and D. Marusic) Let e be a positive integer with $e-1$ a prime power and s a positive integer. Taking the cyclotomy of order e in $\mathbb{F}_{(e-1)^2}$ and the cyclotomic orbits $C_0, \ldots, C_{e-1}$, then

$$S_{i,j} = \begin{cases} C_i & \text{if } i = j, \\ \{0\} \cup C_j & \text{in other case,} \end{cases}$$

is a PSF in the additive group of $\mathbb{F}_{(e-1)^2}$. It generates a DSRG with parameters

$$v = s(e-1)^2, k = s(e-2)+s-1, t = e+s-3, \lambda = e+s-4, \mu = s-1.$$

digraphs

Proposition

(A. Araluze, K. Kutnar, L. Martínez and D. Marusic) Let e be a positive integer with $e-1$ a prime power and s a positive integer. Taking the cyclotomy of order e in $\mathbb{F}_{(e-1)^2}$ and the cyclotomic orbits $C_0, \ldots, C_{e-1}$, then

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is a PSF in the additive group of $\mathbb{F}_{(e-1)^2}$. It generates a DSRG with parameters
$v = s(e-1)^2, k = (s+1)(e-2) + s - 1, t = 2(e-2) + s - 1,$
$\lambda = e + s - 4, \mu = s + 1.$

Contents

uniform multiplicative designs

Definition

(Ryser) A multiplicative design is an incidence structure $\mathfrak{S} = (V, \mathfrak{B}, I)$, with $|V| = v, |\mathfrak{B}| = b$, such that any incidence matrix (with rows and columns indexed by points and blocks, respectively) A is of order $v \geq 3$ and satisfies $AA^t = D + \alpha\alpha^t$ where D is a diagonal matrix with positive entries in the diagonal and $\alpha = (\alpha_1, \ldots, \alpha_v)$ is a vector with real positive entries. If, besides, D is a scalar matrix dI, the design is called a uniform multiplicative design

uniform multiplicative designs

Definition

A uniform multiplicative design is said to be normal if it has the same parameters as its dual, that is, if any incidence matrix satisfies

$$AA^t = A^tA = dI + \alpha\alpha^t \tag{4}$$

It is two-class normal if, besides, the entries of the vector α take only two values α_1 and α_2 and each of one appears e times.

uniform multiplicative designs

Proposition

(Bridges and Mena) If $\mathfrak{S} = (V, \mathfrak{B}, I)$ is a 2-class normal uniform multiplicative design, then d is a square, and one of its two square roots, say σ, is an eigenvalue of A.

uniform multiplicative designs

Proposition

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Definition

It is said that α_1, α_2 and σ are compatible if $\alpha_1\alpha_2 = \sigma^2 - \sigma$ holds.

uniform multiplicative designs

Proposition

(Bridges and Mena) If α_1, α_2 *and* σ *are compatible and if, for the corresponding value of e we have a* $(v', \{k'_1, k'_2\}, \lambda')$*-difference family* $\{B_1, B_2\}$ *on a group G of order* $v' = e$ *with* $k'_2 = \alpha_1\alpha_2, \lambda' = \alpha_1^2, k'_1 + k'_2 - \lambda' = \sigma^2$ *and if A and B are the incidence matrices of the development of* B_1 *and* B_2*, respectively, then*

$$\begin{pmatrix} A & B \\ B^t & J - A^t \end{pmatrix}$$

is an incidence matrix of a normal uniform multiplicative design.

uniform multiplicative designs

Definition

A bi-Cayley uniform multiplicative design is a uniform multiplicative design D whose incidence matrices are adjacency matrices of a bi-Cayley digraph. In this case we will say that (Q, R, S, T) is the symbol of the design D.

uniform multiplicative designs

Definition

Let α_1, α_2 and σ be compatible, and let G be a group of order $\mu + \sigma$. A quadruple (Q, R, S, T) whose elements are subsets of G is a $(\alpha_1, \alpha_2, \sigma)$-multiplicative difference quadruple if the following identities are satisfied in the group ring $\mathbb{Z}[G]$:

1. $RR^{(-1)} + QQ^{(-1)} = d \cdot e + \alpha_1^2 G$
2. $SS^{(-1)} + TT^{(-1)} = d \cdot e + \alpha_2^2 G$
3. $RT^{(-1)} + QS^{(-1)} = \alpha_1 \alpha_2 G$

uniform multiplicative designs

Proposition

A quadruple (Q, R, S, T) *is a* $(\alpha_1, \alpha_2, \sigma)$*-multiplicative difference quadruple iff the adjacence matrix of the bi-Cayley digraph defined by the symbol* (Q, R, S, T) *is an incidence matrix of a uniform multiplicative design corresponding to the compatible* α_1, α_2 *and* σ*.*

uniform multiplicative designs

Proposition

(Conjecture) If G is cyclic and (Q, R, S, T) is a $(\alpha_1, \alpha_2, \sigma)$-multiplicative difference quadruple in G, then there exists a u in G such that

$$-Q = T + u \text{ and } R = -S^c - u.$$

uniform multiplicative designs

THANK YOU VERY MUCH!