Normal Cayley Graphs

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Definition

Let G be a group and $S \subseteq G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The graph $\Gamma(G, S)$ is the Cayley graph of G with connection set S.

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Symmetry is rare. So, if one wishes to obtain examples of a vertex-transitive graphs, the Cayley graph construction is probably the most common method used.

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One can think of a DRR or GRR as being a Cayley (di)graph Γ in which the Cayley (di)graph construction fails to give any additional automorphisms of Γ.

What if one wanted to construct a Cayley digraph Γ of G, but wanted to insist that $\mathrm{Aut}(\Gamma)$ was bigger than just G_L?

Lemma

Let $\Gamma = \Gamma(G, S)$ *be a Cayley digraph of* G *and* $\alpha \in \mathrm{Aut}(G)$*. Then* $\alpha \in \mathrm{Aut}(\Gamma)$ *if and only if* $\alpha(S) = S$*.*

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So one can think of normal Cayley graphs of G as being those for which we construct the graph so that the automorphism group contains some nontrivial automorphisms of G, but for which there are no other graph automorphisms.

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We will primarily be interested in the first approach.

Normal Cayley graphs of $\mathbb{Z}_p$

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Theorem (Burnside, 1901)

Let $G \leq S_p$, p a prime, contain $(\mathbb{Z}_p)_L$. Then $G \leq \mathrm{AGL}(1,p) = \{x \to ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or G is doubly-transitive.

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As $\mathrm{AGL}(1,p) = \mathrm{Aut}(\mathbb{Z}_p) \cdot (\mathbb{Z}_p)_L$ (so $(\mathbb{Z}_p)_L \triangleleft \mathrm{AGL}(1,p)$), with the exception of the complete graph and its complement, every circulant digraph of prime order is normal.

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Theorem (D., D. Witte, 2002)

There are exactly $2p-1$ *transitive p-subgroups* P *of* S_{p^2} *up to conjugation, and all but three have the property that if* $G \leq S_{p^2}$ *with Sylow p-subgroup* P*, then either* $P \triangleleft G$ *or* G *is doubly-transitive.*

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Theorem (D., 2005)

Let P be a transitive p-subgroup of S_{p^k}, p an odd prime, $k \geq 1$, such that every minimal transitive subgroup of P is cyclic. If $G \leq S_{p^k}$ with Sylow p-subgroup P, then either $P \triangleleft G$ or G is doubly-transitive.

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J. Morris has solved this if the other regular subgroup is abelian.

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Theorem (Jones, 1979)

Let $G \leq S_{p^2}$ be transitive with Sylow p-subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then G contains a normal subgroup H such that $H = H_1 \times H_2$ where H_i is a transitive nonabelian simple group of degree p or $H_i = \mathbb{Z}_p$.

Theorem (D. and P. Spiga)

Let $G \leq S_{p^k}$ be transitive with an abelian Sylow p-subgroup P. Then G contains a normal subgroup permutation isomorphic to a direct product of cyclic groups and doubly-transitive nonabelian simple groups with the canonical (coordinate wise) action, where the number of factors in the direct product is equal to the rank of P. Moreover, if $p = 2$, then P is normal in G.

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Theorem (D.)

Every transitive group of prime-power degree p^k that contains a transitive metacyclic Sylow p-subgroup, $p \geq 5$, contains a normal Sylow p-subgroup.

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Suppose you know the groups in S_{G_1} and S_{G_2} that have regular groups G_1 and G_2 as Sylow p-subgroups (both groups of order a power of the prime p). Is it true that the only subgroups of $S_{G_1 \times G_2}$ with Sylow p-subgroup $G_1 \times G_2$ contain a normal subgroup which is a direct product of groups that have Sylow p-subgroup G_1 in S_{G_1} and G_2 in S_{G_2}?

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What if G_2 is cyclic? Can you find conditions on G_1 so that the previous problem is true?

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- Work on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002) give strong constraints on the automorphism group of circulants. Ponomarenko used these results to give a polynomial time algorithm to compute the full automorphism group of a circulant.

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Conjecture

Let G be a finite group of order g which is neither abelian of exponent at least 3 nor generalized dicyclic. Then

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There is also the similar conjecture that "almost all" Cayley digraphs of G are DRR's of G.

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Theorem

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Theorem

Let G be a non-abelian group of prime-power order with no homomorphism onto $\mathbb{Z}_p \wr \mathbb{Z}_p$ (a full Sylow p-subgroup of S_{p^2}). Then almost all Cayley graphs of G are GRR's of G.

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Theorem

Let G be a nilpotent group of odd order g. Let Γ be a random Cayley digraph or Cayley graph of G. In the undirected case, assume additionally that G is not abelian. Then the probability that $\mathrm{Aut}(\Gamma) \neq G$ is less than $(0.91 + o(1))^{\sqrt{g}}$.

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Theorem

Let G be an abelian group of order $g \equiv -1 \pmod 4$. Then, for almost all Cayley graphs Γ of G, $|\mathrm{Aut}(\Gamma)| = 2|G|$.

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Let G be an abelian group of odd prime-power order. Then almost every Cayley graph of G has automorphism group of order $2 \cdot |G|$.

How are these results proven? The basic idea is to show that if $\mathrm{Aut}(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of G contained in $\mathrm{Aut}(\Gamma)$. The connection set of Γ must then be a union of orbits of this automorphism. The number of orbits are then counted, as well as the possible "extra" automorphisms.

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Note that if a G is of prime-power order p^k and a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$ has order greater than p^k, then $N_{\mathrm{Aut}(\Gamma)}(G_L) \neq G_L$.

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Let G be an abelian group of prime-power order p^k. Then almost every Cayley digraph of G that is not a DRR is a normal Cayley digraph of G. In particular,

$$\lim_{p\to\infty} \frac{|NorCayDi(G) - DRR(G)|}{|CayDi(G) - DRR(G)|} = 1.$$

Theorem (D.)

Let G be an abelian group of prime-power order p^k. Then almost every Cayley graph of G that does not have automorphism group of order $2 \cdot |G|$ is a normal Cayley graph of G.

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Note that if $H = K$, a semiwreath product is in fact a wreath product.

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A Cayley graph Γ *of an abelian group* G *is a* **deleted wreath product** *if* $\Gamma = (\Gamma_1 \wr \bar{K}_m) - m\Gamma_1$, *where* Γ_1 *is a Cayley graph of an abelian group of order* $|G|/m$, *and* $m\Gamma_1$ *is* m *vertex-disjoint copies of* Γ_1.

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The preceding conjecture is known to be true if G is cyclic, and is false for some nonabelian groups.

Problem

For an abelian group G, does there exist a natural collection $\mathcal{F}$ of families of Cayley (di)graphs of G and a partial order $\preceq$ on $\mathcal{F}$ such that every Cayley (di)graph of G is contained in some element of $\mathcal{F}$ and if $F_1 \preceq F_2$ and there is no F_3 such that $F_1 \preceq F_3 \preceq F_2$, then almost every Cayley (di)graph of G that is not in F_1 is in F_2?

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Conjecture

Almost every vertex-transitive (di)graph is a Cayley (di)graph.