Regular polytopes with few flags

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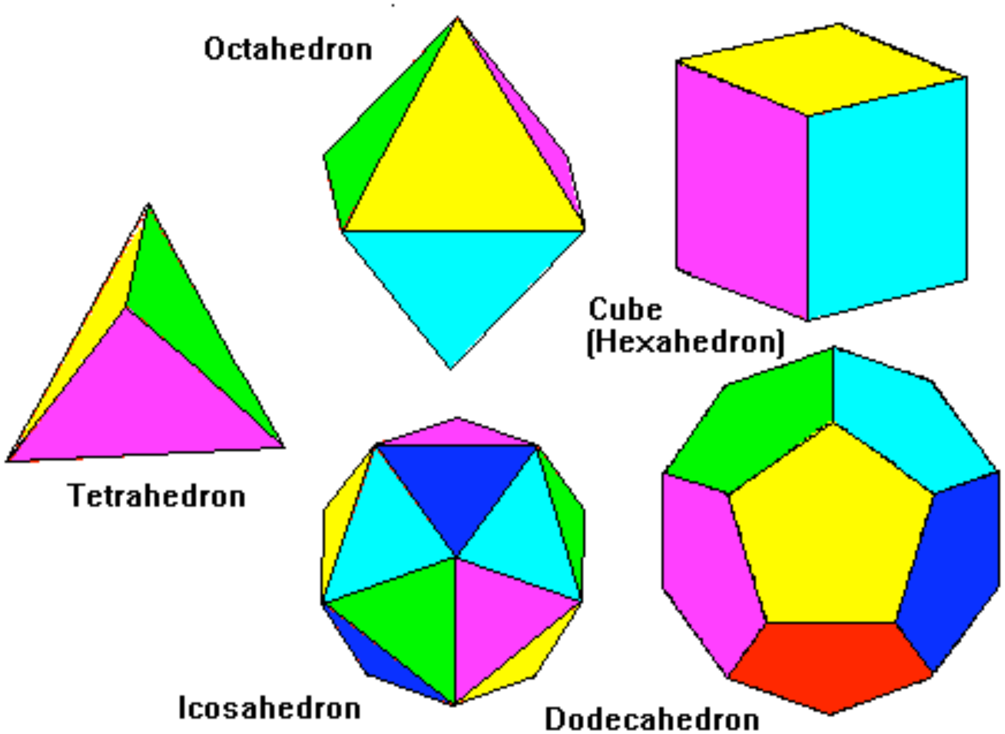
Introduction: Rotary and regular maps

A map M is a 2-cell embedding of a connected graph or multigraph (graph with multiple edges) on a surface, dividing the latter into simply-connected regions called the faces.

Every automorphism of a map M is uniquely determined by its effect on a given flag (incident vertex-edge-face triple), and it follows that $|\text{Aut}\, M| \leq 4|E|$ where E is the edge set.

A map M is regular if $\text{Aut}\, M$ is transitive on flags, and is orientably-regular (or rotary) if the group $\text{Aut}^{\text{o}}M$ of all its orientation-preserving automorphisms is transitive on the ordered edges (arcs) of M. A rotary map that is regular must be reflexible, and otherwise it is chiral.

Platonic solids: regular maps on the sphere

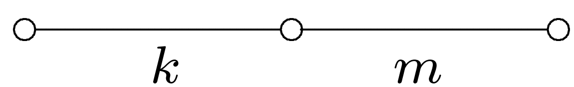


Transitivity, type and triangle groups

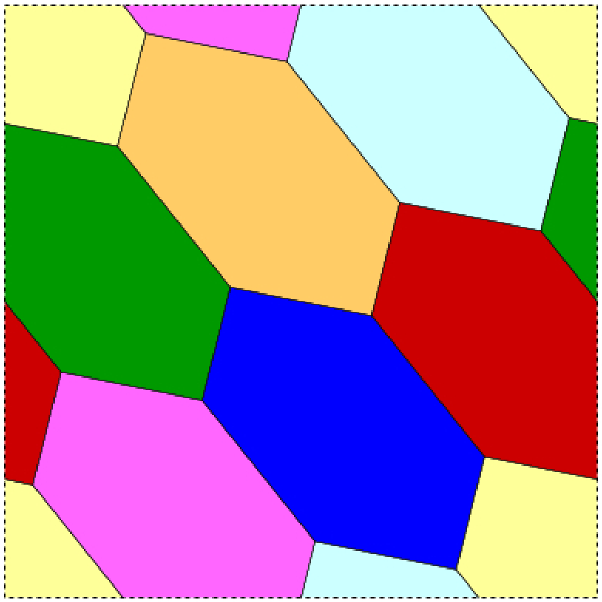
If M is a rotary map, then its underlying graph is vertex-transitive, edge-transitive and face-transitive.

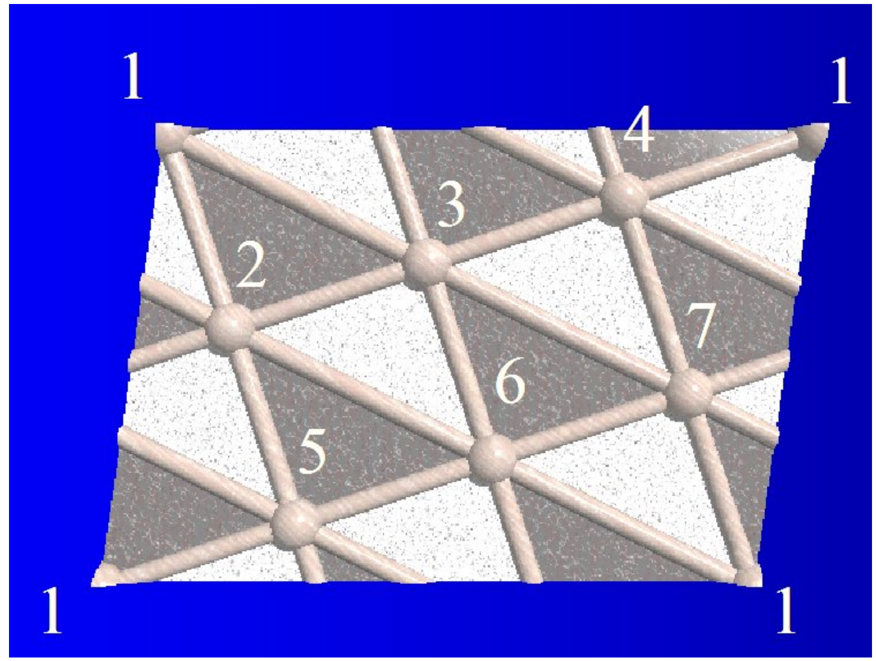
In particular, every face of must have the same number of edges (say k) and every vertex must have the same valency (say m). In this case we say that M has type $\{k, m\}$.

Moreover, Aut M contains elements R and S that act as single-step rotations about a face and an incident vertex, and satisfy the relations $R^k = S^m = (RS)^2 = 1$, which define the $(2, k, m)$ triangle group — a subgroup of index 2 in the $[k, m]$ Coxeter group



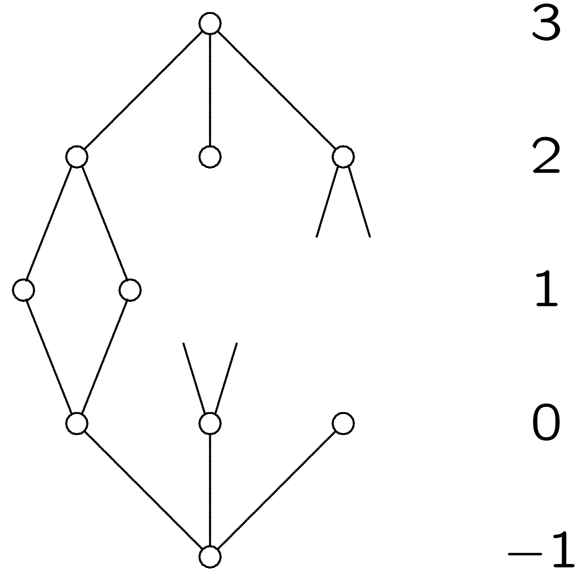
Example: a map of type $\{6, 3\}$ on the torus





This is rotary but chiral, and dual to the triangulation of the torus achieved by embedding the complete graph K_7 as a rotary (but chiral) map of type $\{3, 6\}$.

Polytopes



An *abstract polytope* of rank n is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. For $-1 \leq j \leq n$, elements of $\mathcal{P}$ of rank j are called the *j-faces*, and a typical j-face is denoted by F_j.

This poset must satisfy certain combinatorial conditions which generalise the properties of geometric polytopes.

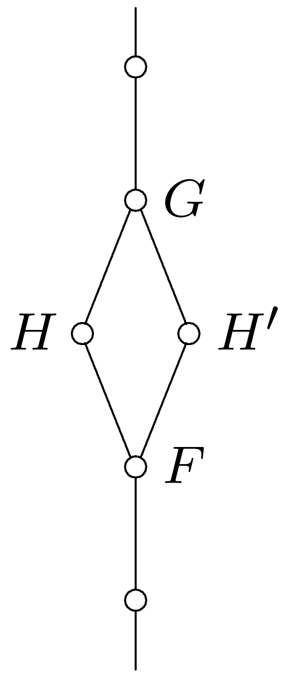
We require that $\mathcal{P}$ have a smallest (-1)-face F_{-1}, and a greatest n-face F_n, and that each maximal chain (or *flag*) of $\mathcal{P}$ has length $n+2$, e.g. $F_{-1} - F_0 - F_1 - F_2 - ... - F_{n-1} - F_n$.

The faces of rank $0, 1$ and $n-1$ are called the *vertices*, *edges* and *facets* of the polytope, respectively.

Two flags are called *adjacent* if they differ by just one face.

We require that $\mathcal{P}$ is *strongly flag-connected*, that is, any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $(\Phi =)\,\Phi_0,\, \Phi_1, \ldots,\, \Phi_k\,(= \Psi)$ such that each two successive faces Φ_{i-1} and Φ_i are adjacent, and $\Phi \cap \Psi \subseteq \Phi_i$ for all i.

Finally, we require the following homogeneity property, which is often called the *diamond condition* :



Whenever $F \leq G$, with *rank*$(F) = j - 1$ and *rank*$(G) = j + 1$, there are exactly two faces H of rank j such that $F \leq H \leq G$

[e.g. if $F =$ vertex, $H =$ 2-face, then $\exists$ two incident edges].

A little history

Regular and chiral maps

Heffter (1898), Brahana (1927), Coxeter (1948), Sherk, Garbe, Wilson, Jones, Singerman, ...

Convex geometric polytopes

Various (e.g. Coxeter, Grünbaum, et al)

'Non-spherical' polytopes

Grünbaum (1970s)

Incidence polytopes

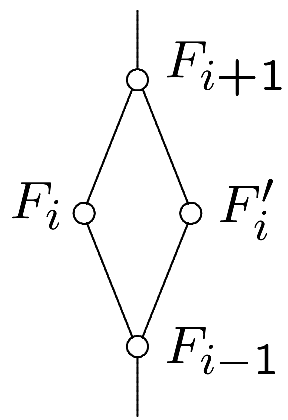
Danzer & Schulte (1983)

Regular & chiral polytopes

Weber & Seifert (1933), Coxeter, Schulte, Weiss, Nostrand, Hartley, Leemans, Hubard, Pellicer, ...

An *automorphism* of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$. A polytope $\mathcal{P}$ is *regular* if the automorphism group Aut($\mathcal{P}$) is transitive on the flags of $\mathcal{P}$.

When $\mathcal{P}$ is regular, Aut($\mathcal{P}$) can be generated by n involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$, where each ρ_i maps a given *base flag* Φ to the adjacent flag Φ^i (differing from Φ only in its i-face).



... ρ_i swaps F_i with F_i'

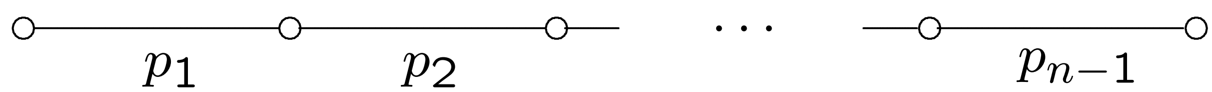
Thus Aut($\mathcal{P}$) a quotient of the Coxeter group $[p_1, \ldots, p_{n-1}]$, where $p_i = o(\rho_{i-1}\rho_i)$ for $1 \leq i < n$.

Coxeter groups

The Coxeter group $[p_1, \ldots, p_{n-1}]$ is the group generated by elements $x_1, x_2, \ldots, x_n$ subject to the defining relations:

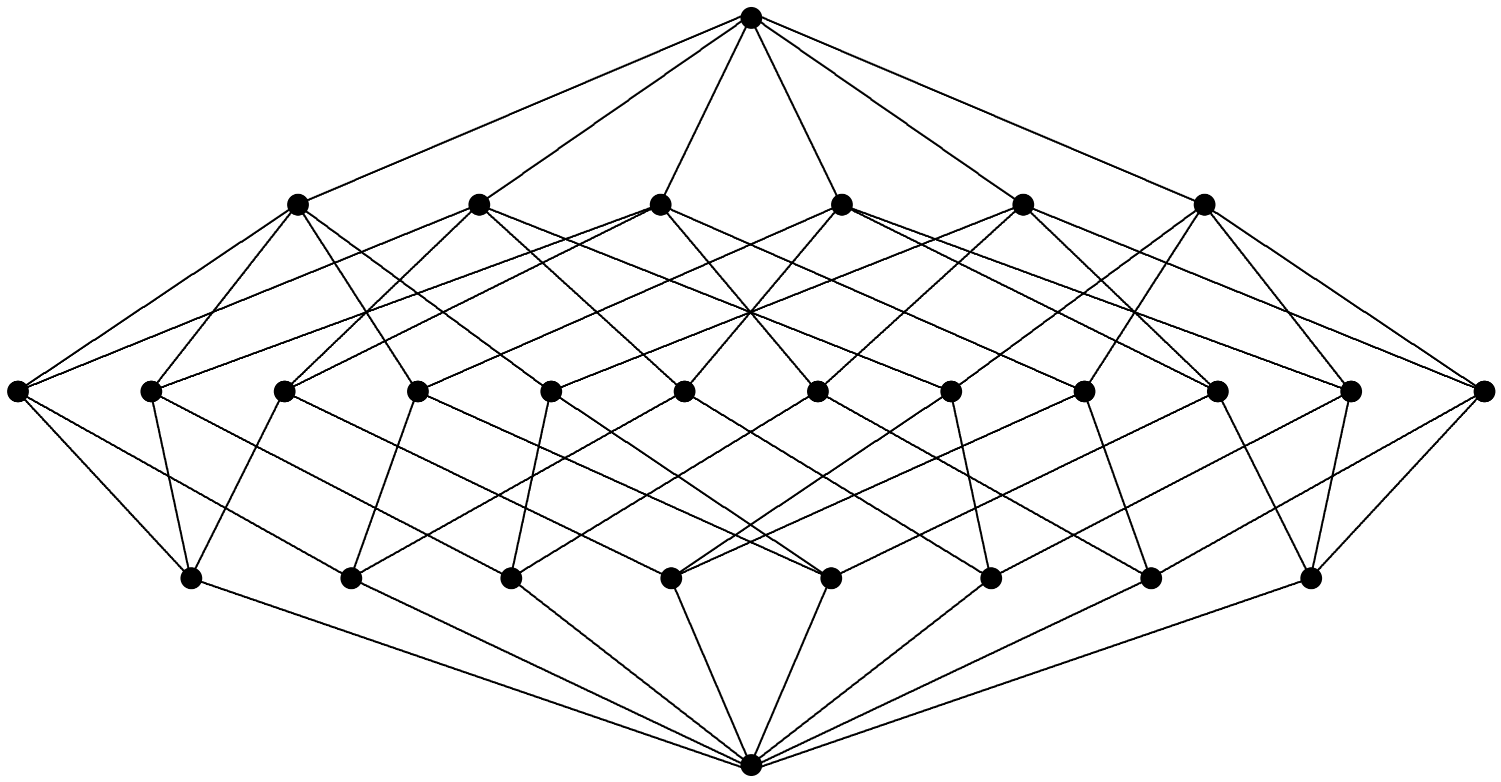
- $x_i{}^2 = 1$ [Generators are involutions]
- $(x_{i-1}x_i)^{p_i} = 1$ [Orders of products of consecutive pairs]
- $(x_i x_j)^2 = 1$ for $|i-j| > 1$ [Non-consecutive pairs]

This presentation can be represented by the Dynkin diagram



The n–1 products $x_{i-1}x_i$ generate the 'even word' subgroup (which has index 2, and relates to polytope orientability).

Example: the cube (a 3-polytope of type $\{3, 4\}$)



$\text{Aut}(Q_3) \cong S_4 \times C_2$ is a quotient of the $[3, 4]$ Coxeter group

Infinite families of regular polytopes

There are many families of regular polytopes, including these:

- Regular n-simplex, of type $\{3, \overset{n-1}{\ldots}, 3\}$, autom group S_{n+1}
- Cross polytope (or n-orthoplex), of type $\{3, \overset{n-2}{\ldots}, 3, 4\}$
- n-dimensional cubic honeycomb, of type $\{4, 3, \overset{n-2}{\ldots}, 3, 4\}$

[Note: The situation for 'chiral' polytopes is quite different]

Example family: The regular n-simplices

Let $X = \{1, 2, \ldots, n+1\}$, and form an abstract n-polytope $\mathcal{P}$ by taking all the subsets of X, ordered by inclusion: vertices are the singletons $\{i\}$, edges are the pairs $\{i, j\}$, etc. Note that the number of flags is $(n+1)n(n-1)\ldots 1 = (n+1)!$.

Each inclusion $A \leq B$ with $|B| = |A| + 2$ (say $B = A \cup \{x, y\}$) has two refinements $A \leq A \cup \{x\} \leq B$ and $A \leq A \cup \{y\} \leq B$, so the diamond condition holds.

Moreover, the transposition $\tau = (x, y)$ in S_{n+1} takes any flag of the form $\phi - F_0 - \cdots - A - A \cup \{x\} - B - \cdots - F_n - X$ to the flag $\phi - F_0 - \cdots - A - A \cup \{y\} - B - \cdots - F_n - X$, and hence the symmetric group S_{n+1} acts regularly on flags.

This is the regular n-simplex (known in geometry/topology).

Some observations

The automorphism group S_{n+1} is a quotient of the Coxeter group $[3, 3, \ldots, 3]$, via an epimorphism taking the generators to the transpositions $\tau_i = (i, i+1)$ for $1 \le i \le n$.

Note that $\tau_i\tau_{i+1} = (i, i+1)(i+1, i+2) = (i, 1+2, i+1)$ which has order 3, while $\tau_i\tau_j = (i, i+1)(j, j+1)$ has order 2 when $|i-j| > 1$. Hence the regular n-simplex has type $\{3, 3, \ldots, 3\}$.

Question: What happens if the 'type' contains a 2?

• If $(x_i x_{i+1})^2 = 1$ then the Coxeter group is a direct product of the Coxeter groups $\langle x_0, \ldots, x_i \rangle$ and $\langle x_{i+1}, \ldots, x_n \rangle$

• If the type is $\{2, \overset{n-1}{\ldots}, 2\}$, then $\mathrm{Aut}(\mathcal{P})$ is generated by n commuting involutions, so $\mathcal{P}$ has just two i-faces for each i, and the number of flags is 2^n — not very interesting.

Construction of new examples

Let G be a finite group generated by involutions $g_1, g_2, \ldots, g_n$.

Let $H_i = \langle g_1, g_2, \ldots, g_{i-1}, \widehat{g_i}, g_{i+1}, \ldots, g_n \rangle$ for $1 \le i \le n$, and define a ranked poset $\mathcal{P} = \mathcal{P}(G)$ as follows:

- Take as j-faces the right cosets of H_{j+1} in G, for $0 \le j < n$
- Say that $H_j a \le H_k b$ if and only if $H_j a \cap H_k b \ne \phi$, for $j \le k$

Then G acts naturally (by right multiplication) on this poset as a flag-transitive group of order-preserving bijections, and $\mathcal{P}(G)$ is a polytope iff G satisfies if the intersection property

$$\langle\, g_i \,|\, i \in I \,\rangle \cap \langle\, g_j \,|\, j \in J \,\rangle = \langle\, g_k \,|\, i \in I \cap J \,\rangle \quad \forall\, I, J \subseteq \{1, 2, \ldots, n\}$$

— which is precisely equivalent to the diamond condition.

Recent question [Daniel Pellicer (Oaxaca, June 2010)]

For each $n \geq 3$, what are the regular n-polytopes with the smallest numbers of flags? Call the smallest number M_n.

Answering this question is equivalent to finding the smallest good quotients of n-generator Coxeter groups $[p_1, \ldots, p_{n-1}]$ — with 'good' meaning that the orders of the generators x_i and their pairwise products $x_i x_j$ are preserved, and the intersection property holds.

Also we may assume that $p_i > 2$ for all i (for otherwise the question is not very interesting).

Small ranks

These results achievable by computation (using MAGMA):

Rank n	$(n+1)!$	Min # flags M_n	Types of polytopes achieving minimum
2	6	6	$\{3\}$
3	24	24	$\{3,3\}$, $\{3,4\}$, $\{4,3\}$
4	120	96	$\{4,3,4\}$
5	720	432	$\{3,6,3,4\}$, $\{4,3,6,3\}$
6	5040	1728	$\{4,3,6,3,4\}$

Surprisingly(?), the minimum type is not always $\{3,3,\ldots,3\}$

Is there a pattern evident here? Are extensions possible?

Two new families

MAGMA computations give also defining presentations for the automorphism groups of small examples. Patterns in these give rise to constructions for two infinite families:

- A regular n-polytope of type $\{4, 3, 6, 3, 6, 3, 6, ..., 3, 6, 3\}$, with $8 \cdot 3^{(n-1)/2} \cdot 6^{(n-3)/2}$ flags, for every **odd** $n > 2$
- A regular n-polytope of type $\{4, 3, 6, 3, 6, 3, 6, ..., 3, 6, 3, 4\}$, with $32 \cdot 3^{(n-2)/2} \cdot 6^{(n-4)/2}$ flags, for every **even** $n > 2$.

Thus $M_n \leq \begin{cases} 24 \cdot 18^{(n-3)/2} & \text{for } n \text{ odd} \\ 96 \cdot 18^{(n-4)/2} & \text{for } n \text{ even} \end{cases}$

Are these the best?

We could call a regular n-polytope tight if the number of its flags is $2p_1p_2...p_{n-1}$, where $\{p_1, p_2, ..., p_{n-1}\}$ is its type.

All the polytopes constructed in the families above (of types $\{4, 3, 6, 3, 6, ..., 3, 6, 3\}$ and $\{4, 3, 6, 3, 6, ..., 3, 6, 3, 4\}$) are tight.

Can we prove these give the smallest numbers of flags for all n? or are there too many ‘6’s in the type?

In the course of trying to prove they were the best, another family popped into view ...

Tight regular polytopes of type {4,4,...,4}

There exist regular polytopes of types {4,4}, {4,4,4} and {4,4,4,4}, with 32, 128 and 512 flags. Closer inspection of these (and their automorphism groups) gives a new family:

For every $n > 2$, take the Coxeter group $[4,\overset{n-1}{\ldots},4]$, with n involutory generators $x_1, x_2, ..., x_n$, and add relations of the form $[(x_{i-1}x_i)^2, x_j] = 1$ to make the squares $(x_{i-1}x_i)^2$ all central. This gives a group G whose centre $Z(G)$ is generated by the $n-1$ involutions $(x_{i-1}x_i)^2$.

In particular, $Z(G)$ and $G/Z(G)$ are elementary abelian, of orders 2^{n-1} and 2^n, so G has order $2^{2n-1} = 2 \cdot 4^{n-1}$. Also the intersection property holds, so G is the automorphism group of a tight regular n-polytope of type $\{4, 4, ..., 4\}$.

Improved upper bounds on M_n

Tight polytopes of type $\{4, ..., 4\}$ give $M_n \leq 2 \cdot 4^{n-1}$ for all n.

This is better than our earlier upper bound of $24 \cdot 18^{(n-3)/2}$ for n odd, and $96 \cdot 18^{(n-4)/2}$ for n even, whenever $n > 8$.

Question: Is the bound $M_n \leq 2 \cdot 4^{n-1}$ sharp for all $n > 8$?

Question: We know M_3 to M_6. What are M_7 and M_8?

Key observation

Suppose $\mathcal{P}$ is a regular n-polytope, of type $\{p_1, \ldots, p_{n-1}\}$.

Each of the sections of $\mathcal{P}$ is also a regular polytope.
In particular, if A and B are i- and j- faces of $\mathcal{P}$ with $A \le B$, then the section $[A, B] = \{F \in \mathcal{P} : A \le F \le B\}$ is a regular $(j-i-1)$-polytope. Its automorphism group is $\langle \rho_{i+1}, \ldots, \rho_{j-1} \rangle$ where $\rho_0, \rho_1, \ldots, \rho_{n-1}$ are the Coxeter generators for Aut($\mathcal{P}$).

Next, for any k, let $L_k = \langle \rho_0, \ldots, \rho_k \rangle$ and $R_k = \langle \rho_{k-1}, \ldots, \rho_{n-1} \rangle$.
By the intersection property, $L_k \cap R_k = \langle \rho_{k-1}, \rho_k \rangle \cong D_{p_k}$

so $|\text{Aut}(\mathcal{P})| \ge |L_k R_k| = |L_k||R_k|/|L_k \cap R_k| = |L_k||R_k|/|D_{p_k}|$.

It follows that $M_n \ge \frac{M_{k+1} M_{n-k+1}}{2p_k}$ for $1 \le k \le n-1$.

As $L_k \cap R_{k+1} = \langle \rho_k \rangle$, also $M_n \ge \frac{M_{k+1} M_{n-k}}{2}$ for $1 \le k \le n-2$.

Application

Suppose $M_n = 2 \cdot 4^{n-1}$ for all n in the range $k < n \leq 2k$.

Then

$$M_{2k+1} \geq \frac{M_{k+1} M_{k+1}}{2} = \frac{(2 \cdot 4^k)^2}{2} = 2 \cdot 4^{2k}$$

and similarly

$$M_{2k+2} \geq \frac{M_{k+1} M_{k+2}}{2} = \frac{(2 \cdot 4^k)(2 \cdot 4^{k+1})}{2} = 2 \cdot 4^{2k+1}$$

and so $M_n = 2 \cdot 4^{n-1}$ for all n in the range $k < n \leq 2k + 2$.

This gives a basis for induction. We just have to find a k ...

Finding M_n for small $n \geq 7$

With the help of the `LowIndexNormalSubgroups` algorithm in MAGMA (applied to Coxeter groups), we can find:

- all regular 3-polytopes with up to 100 flags
- all regular 4-polytopes with up to 300 flags
- all regular 5-polytopes with up to 900 flags
- all regular 6-polytopes with up to 2700 flags.

Then multiple applications of the intersection property show:

- the only regular 7-polytopes with fewer than $2 \cdot 4^6$ flags have type $\{4, 3, 6, 3, 6, 3\}$ or $\{3, 6, 3, 6, 3, 4\}$ (and 7776 flags)
- the only regular 8-polytope with fewer than $2 \cdot 4^7$ flags has type $\{4, 3, 6, 3, 6, 3, 4\}$ (and 31104 flags), and
- for $9 \leq n \leq 16$, the smallest regular n-polytope is a tight one of type $\{4, 4, ..., 4\}$ (with $2 \cdot 4^{n-1}$ flags).

Example: $n = 9$

Suppose there is a regular 9-polytope of type $\{p_1, p_2, ..., p_8\}$ with fewer than $2 \cdot 4^8 = 131072$ flags.

By taking the dual if necessary, we can assume that some 5-face F has fewer than $2 \cdot 4^4 = 512$ flags. Then F must have exactly 432 flags and have type $\{3, 6, 3, 4\}$ or $\{4, 3, 6, 3\}$, and its co-5-face must have at most 606 flags, with its type $\{p_5, p_6, p_7, p_8\}$ coming from a known list.

Then the given 9-polytope has type $\{3, 6, 3, 4, p_5, p_6, p_7, p_8\}$ or $\{4, 3, 6, 3, p_5, p_6, p_7, p_8\}$, but from our lists of small regular 6-polytopes we find no 6-section of type $\{3, 4, p_5, p_6, p_7\}$ or $\{6, 3, p_5, p_6, p_7\}$ small enough to give fewer than $2 \cdot 4^8$ flags.

Theorem

For $n \geq 9$, the smallest regular n-polytopes are the tight polytopes of type $\{4, \overset{n-1}{\ldots}, 4\}$, with $2 \cdot 4^{n-1}$ flags.

For $n \leq 8$, the smallest have the following parameters:

n	M_n	Type(s)
2	6	$\{3\}$
3	24	$\{3, 3\}$, $\{3, 4\}$ (and dual $\{4, 3\}$)
4	96	$\{4, 3, 4\}$
5	432	$\{3, 6, 3, 4\}$ (and dual $\{4, 3, 6, 3\}$)
6	1728	$\{4, 3, 6, 3, 4\}$
7	7776	$\{3, 6, 3, 6, 3, 4\}$ (and dual $\{4, 3, 6, 3, 6, 3\}$)
8	31104	$\{4, 3, 6, 3, 6, 3, 4\}$.