Leonard triples associated with hypercubes and their antipodal quotients

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August 4, 2010

Definition of a Leonard pair

Let $\mathbb{K}$ denote a field and let V be a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair on V of diameter d* we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ which satisfy the conditions (i), (ii) below.

(i) There exists a basis $\{v_i\}_{i=0}^d$ for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal,

(ii) There exists a basis $\{v_i^*\}_{i=0}^d$ for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

Definition of a Leonard triple

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(i) There exists a basis $\{v_i\}_{i=0}^d$ for V with respect to which the matrix representing A is diagonal and the matrix representing A^*, A^ε are irreducible tridiagonal,

(ii) There exists a basis $\{v_i^*\}_{i=0}^d$ for V with respect to which the matrix representing A^* is diagonal and the matrix representing A^ε, A is irreducible tridiagonal,

(ii) There exists a basis $\{v_i^\varepsilon\}_{i=0}^d$ for V with respect to which the matrix representing A^ε is diagonal and the matrix representing A, A^* is irreducible tridiagonal.

Definition of distance-regular graphs

Let Γ be a graph with vertex set $\mathbf{X}$ and edge set $\mathbf{E}$. Given two vertices $x, y \in \mathbf{X}$ we define the *distance* between x and y to be the length of the shortest path from x to y, denoted $\partial(x, y)$. The diameter of Γ, denoted d, is the longest distance between any two points in $\mathbf{X}$.
For $0 \leq i \leq d$, we say the ith distance matrix, A_i is the matrix indexed by $\mathbf{X}$ such that the xy entry is 1 if $\partial(x, y) = i$ and 0 otherwise. A_1 is called the *adjacency matrix*, also denoted A.
The graph Γ is said to be *distance-regular* if $A_i A_j$ is a linear combination of the distance matrices for $0 \leq i, j \leq d$. Note this means A_i is a degree-i polynomial in A for $0 \leq i \leq d$.

The Subsonstituent algebra T

A is diagonalizable, with $d+1$ eigenvalues. Let $\theta_0, \theta_1, \ldots, \theta_d$ be an ordering of the eigenvalues and let $E_0, E_1, \ldots, E_d$ be the corresponding primitive idempotents.
Fix a vertex x and, for $0 \leq i \leq d$ define E_i^* be the diagonal matrix with yy entry 1 when $\partial(x, y) = i$ and 0 otherwise. The Subconstituent algebra of Γ with respect to x denoted $T(x)$, is the algebra generated by $A, E_0^*, E_1^*, \ldots, E_d^*$.
If V is the vector space indexed by **X** then
$AE_i^* V = E_{i-1}^* V + E_i^* V + E_{i+1}^* V$ for $0 \leq i \leq d$ (with $E_{-1}^* = E_{d+1}^* = 0$).

The Q-polynomial property

Given our ordering $\theta_0, \theta_1, \ldots, \theta_d$ of eigenvalues and our vertex $x \in \mathbf{X}$, we define the *dual distance matrices* $A_0^*, A_1^*, \ldots, A_d^*$ to be diagonal matrices with yy entry equal to $m_i(E_i)_{xy}$ where m_i is the multiplicity of θ_i for $0 \leq i \leq d$. Note that, for $0 \leq i \leq d$, A_i^* has $E_0^*, E_1^*, \ldots, E_d^*$ as its primitive idempotents.

The ordering $\theta_0, \theta_1, \ldots, \theta_d$ of eigenvalues for A is said to be Q-polynomial if $A_1^* E_i V = E_{i-1}V + E_i V + E_{i+1} V$ for $0 \leq i \leq d$ (with $E_{-1} = E_{d+1} = 0$). In this case A_1^* is abbreviated A^* and called the *dual adjacency matrix*

Γ is said to be Q-polynomial if such an ordering of eigenvalues exists.

Leonard pairs and Q-polynomial drg's

In a Q-polynomial distance-regular graph the actions of A, A^* on an irreducible $T(x)$-module will always form a Leonard pair or a Tridiagonal pair (See Terwilliger's talk). There is always one $T(x)$-module with the same diameter as the graph. This is called the *standard module*. In this module, the actions of A, A^* always form a Leonard Pair.

We will now look at specific Q-polynomial distance-regular graphs where these Leonard pairs extend to Leonard triples.

Hypercubes

Given nonnegative integer d, the *hypercube of diameter* d, abbreviated Q_d has vertex set consisting of binary strings of length d, where two vertices are adjacent whenever the corresponding strings differ in exactly one entry. This is a distance-regular graph with Q-polynomial ordering $\{d - 2i\}_{i=0}^{d}$ of eigenvalues. Miklavič (2008) showed that you can produce an *imaginary adjacency matrix* A^{ε} for the hypercube, defined by $A^{\varepsilon} = [A, A^*]\frac{1}{2\mathbf{i}}$.

$\mathfrak{sl}_2(\mathbb{C})$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the complex Lie algebra with generators X, Y, Z and relations

$$[X, Y] = 2\mathbf{i}Z, \quad (1)$$

$$[Y, Z] = 2\mathbf{i}X, \quad (2)$$

$$[Z, X] = 2\mathbf{i}Y \quad (3)$$

The actions of X, Y, Z on a finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module will form a leonard triple. In $T(x)$ for Q_d, the matrices A, A^*, A^ε satisfy these relations, so they act as a Leonard triple on every irreducible $T(x)$-module.

Leonard triples and $\mathfrak{sl}_2(\mathbb{C})$-modules

Let d be a nonnegative integer. Then there is one $(d+1)$-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module up to isomorphism. There exist bases $\{v_i\}_{i=0}^d, \{v_i^*\}_{i=0}^d, \{v_i^\varepsilon\}_{i=0}^d$ such that the matrix representing X with respect to $\{v_i\}_{i=0}^d$, Y with respect to $\{v_i^*\}_{i=0}^d$ and Z with respect to $\{v_i^\varepsilon\}_{i=0}^d$ is

$$\begin{pmatrix} d & & & & & & \\ & d-2 & & & & & \\ & & d-4 & & & & \\ & & & \ddots & & & \\ & & & & 4-d & & \\ & & & & & 2-d & \\ & & & & & & -d \end{pmatrix},$$

Leonard triples and $\mathfrak{sl}_2(\mathbb{C})$-modules, continued

the matrix representing X with respect to $\{v_i^*\}_{i=0}^d$, Y with respect to $\{v_i^\varepsilon\}_{i=0}^d$ and Z with respect to $\{v_i\}_{i=0}^d$ is

$$\begin{pmatrix} 0 & d & & & & & \\ 1 & 0 & d-1 & & & & \\ & 2 & 0 & d-2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & d-2 & 0 & 2 & \\ & & & & d-1 & 0 & 1 \\ & & & & & d & 0 \end{pmatrix},$$

Leonard triples and $\mathfrak{sl}_2(\mathbb{C})$-modules, concluded

the matrix representing X with respect to $\{v_i^\varepsilon\}_{i=0}^d$, Y with respect to $\{v_i\}_{i=0}^d$ and Z with respect to $\{v_i^*\}_{i=0}^d$ is

$$\begin{pmatrix} 0 & d\mathbf{i} & & & & & \\ -\mathbf{i} & 0 & (d-1)\mathbf{i} & & & & \\ & -2\mathbf{i} & 0 & (d-2)\mathbf{i} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & (2-d)\mathbf{i} & 0 & 2\mathbf{i} & \\ & & & & (1-d)\mathbf{i} & 0 & \mathbf{i} \\ & & & & & -d\mathbf{i} & 0 \end{pmatrix}.$$

$\mathcal{A}$

We can also obtain Leonard triples from the following algebra.
Let $\mathcal{A}$ be the unital associative complex algebra with generators x, y, z and relations

$$xy + yx = 2z, \tag{4}$$

$$yz + zy = 2x, \tag{5}$$

$$zx + xz = 2y \tag{6}$$

This comes from a nonstandard quantum deformation of $\mathfrak{sl}_2(\mathbb{C})$ taken when $q = -1$.
The actions of x, y, z on a finite-dimensional irreducible $\mathcal{A}$-module will form a Leonard triple.

Bipartite Leonard triples and $\mathcal{A}$-modules

Let d be a nonnegative even integer. Then there is one bipartite $(d+1)$-dimensional irreducible $\mathcal{A}$-module up to isomorphism. There exist bases $\{v_i\}_{i=0}^d, \{v_i^*\}_{i=0}^d, \{v_i^\varepsilon\}_{i=0}^d$ such that the matrix representing x with respect to $\{v_i\}_{i=0}^d$, y with respect to $\{v_i^*\}_{i=0}^d$ and z with respect to $\{v_i^\varepsilon\}_{i=0}^d$ is

$$\begin{pmatrix} d & & & & & & \\ & 2-d & & & & & \\ & & d-4 & & & & \\ & & & \ddots & & & \\ & & & & 4-d & & \\ & & & & & d-2 & \\ & & & & & & -d \end{pmatrix},$$

Bipartite Leonard triples and $\mathcal{A}$-modules, continued

the matrix representing x with respect to $\{v_i^*\}_{i=0}^d$, y with respect to $\{v_i^\varepsilon\}_{i=0}^d$ and z with respect to $\{v_i\}_{i=0}^d$ is

$$\begin{pmatrix} 0 & d & & & & & \\ 1 & 0 & d-1 & & & & \\ & 2 & 0 & d-2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & d-2 & 0 & 2 & \\ & & & & d-1 & 0 & 1 \\ & & & & & d & 0 \end{pmatrix},$$

Bipartite Leonard triples and $\mathcal{A}$-modules, concluded

the matrix representing x with respect to $\{v_i^{\varepsilon}\}_{i=0}^d$, y with respect to $\{v_i\}_{i=0}^d$ and z with respect to $\{v_i^*\}_{i=0}^d$ is

$$\begin{pmatrix} 0 & d & & & & & \\ 1 & 0 & 1-d & & & & \\ & -2 & 0 & d-2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 2-d & 0 & 2 & \\ & & & & d-1 & 0 & -1 \\ & & & & & -d & 0 \end{pmatrix}.$$

Almost bipartite Leonard triples and $\mathcal{A}$-modules

Let d be a nonnegative odd integer with $d = 2h + 1$. Then there are four almost bipartite $(h+1)$-dimensional irreducible $\mathcal{A}$-module up to isomorphism determined by a choice of $\varepsilon, \delta \in \{\pm 1\}$. There exist bases $\{v_i\}_{i=0}^d, \{v_i^*\}_{i=0}^d, \{v_i^\varepsilon\}_{i=0}^d$ such that the matrix representing x with respect to $\{v_i\}_{i=0}^d$, y with respect to $\{v_i^*\}_{i=0}^d$ and z with respect to $\{v_i^\varepsilon\}_{i=0}^d$ is

$$\varepsilon \begin{pmatrix} (-1)^h d & & & & & & \\ & (-1)^h(2-d) & & & & & \\ & & (-1)^h(d-4) & & & & \\ & & & \ddots & & & \\ & & & & 5 & & \\ & & & & & -3 & \\ & & & & & & 1 \end{pmatrix},$$

Almost bipartite Leonard triples and $\mathcal{A}$-modules, continued

the matrix representing x with respect to $\{v_i^*\}_{i=0}^d$, y with respect to $\{v_i^\varepsilon\}_{i=0}^d$ and z with respect to $\{v_i\}_{i=0}^d$ is

$$\delta(-1)^h \begin{pmatrix} 0 & d & & & & & \\ 1 & 0 & d-1 & & & & \\ & 2 & 0 & d-2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & h-2 & 0 & h+3 & \\ & & & & h-1 & 0 & h+2 \\ & & & & & h & h+1 \end{pmatrix},$$

Almost bipartite Leonard triples and $\mathcal{A}$-modules, concluded

the matrix representing x with respect to $\{v_i^{\varepsilon}\}_{i=0}^{d}$, y with respect to $\{v_i\}_{i=0}^{d}$ and z with respect to $\{v_i^{*}\}_{i=0}^{d}$ is $\delta\varepsilon$ times

$$\begin{pmatrix} 0 & d & & & & \\ 1 & 0 & 1-d & & & \\ & -2 & \ddots & \ddots & & \\ & & \ddots & 0 & (-1)^{h-2}(h+3) & \\ & & & (-1)^{h-2}(h-1) & 0 & (-1)^{h-1}(h+2) \\ & & & & (-1)^{h-1}h & (-1)^{h}h+1 \end{pmatrix}.$$

From $\mathfrak{sl}_2$-modules to $\mathcal{A}$-modules

Let V be a finite-dimensional $\mathfrak{sl}_2$-module. Define the operator

$$p = \exp\left(\frac{\mathbf{i}X - Y}{2}\right)\exp\left(\frac{\mathbf{i}X + Y}{2}\right)\exp\left(\frac{\mathbf{i}X - Y}{2}\right).$$

Then $pX = Xp$, $pY = -Yp$, $pZ = -Zp$ and p^2 is central and acts as $(-1)^d$ on an irreducible submodule of diameter d.
Let k be a central operator on V that acts as 1 on even-diameter submodules and $\mathbf{i}$ on odd-diameter submodules. and let $s = pk$.
Then $sX = Xs$, $sY = -Ys$, $sZ = -Zs$ and $s^2 = I$.

From $\mathfrak{sl}_2$-modules to $\mathcal{A}$-modules, continued.

This means that V can be given an $\mathcal{A}$-module structure with x, y, z acting as $sX, Y, s\mathbf{i}Z$ respectively. The same is true if you replace s with $-s$.
If d is even and V is a $(d+1)$-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, then V is irreducible as an $\mathcal{A}$-module and you will get the bipartite Leonard pairs.
If d is odd and V is a $(d+1)$-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, then V is a direct sum of two almost bipartite $\mathcal{A}$-modules.

The alternate Q-polynomial structure for Q_d with d even

We can use the above result to give $\mathcal{A}$-module structures to the subconstituent algebras of hypercubes and related graphs.

When d is even, the hypercube Q_d has two Q-polynomial structures. The other Q-polynomial ordering of eigenvalues is $\{(-1)^i(d-2i)\}$. With respect to this ordering, let $\{a_i\}_{i=0}^d, \{a_i^*\}_{i=0}^d, \{e_i\}_{i=0}^d, \{e_i^*\}_{i=0}^d$ denote the distance matrices, dual distance matrices, primitive idempotents and dual primitive idempotents respectively.

With respect to the ordering $\{d-2i\}_{i=0}^d$, let $\{A_i\}_{i=0}^d, \{A_i^*\}_{i=0}^d, \{E_i\}_{i=0}^d, \{E_i^*\}_{i=0}^d$ denote the distance matrices, dual distance matrices, primitive idempotents and dual primitive idempotents respectively.

The alternate Q-polynomial structure, continued

Then we have, for $0 \leq i \leq d$, $a_i = A_i$, $e_i^* = E_i^*$, but

$$A_i^* = \begin{cases} a_i^* & i \text{ even} \\ a_{d-i}^* & i \text{ odd,} \end{cases}$$

and

$$E_i = \begin{cases} e_i & i \text{ even} \\ e_{d-i} & i \text{ odd.} \end{cases}$$

So $a^* = A_{d-1}^*$.

The alternate Q-polynomial structure, concluded

Because, A, A^*, A^ε act as Y, X, Z, we have that $T(x)$ is an $\mathcal{A}$-module where y, x, z act as $A, (-1)^{\frac{d}{2}} sA^*, (-1)^{\frac{d}{2}} s\mathbf{i}A^\varepsilon$. We also have that for $0 \leq i \leq d$

$$(-1)^{\frac{d}{2}} sA_i^* = \begin{cases} a_i^* & i \text{ even} \\ a_{d-i}^* & i \text{ odd}, \end{cases}$$

so $(-1)^{\frac{d}{2}} sA = a$. This means the Leonard pairs from the alternate Q-polynomial structure of Q_d also extend to Leonard triples, with alternate imaginary adjacency matrix $a^\varepsilon = (-1)^{\frac{d}{2}} s\mathbf{i}A^\varepsilon$.

The antipodal quotient of Q_d with d odd

When d is odd $(2h+1)$, the hypercube Q_d only has one Q-polynomial structure. Instead we look at $\tilde{Q}_d$, the antipodal quotient of Q_d. This is a distance-regular graph with $\{(-1)^i(d-2i)\}_{i=0}^{h}$ a Q-polynomial ordering of its eigenvalues.
Let $\{a_i\}_{i=0}^{h}$,$\{a_i^*\}_{i=0}^{h}$,$\{e_i\}_{i=0}^{h}$,$\{e_i^*\}_{i=0}^{h}$ denote the distance matrices, dual distance matrices, primitive idempotents and dual primitive idempotents respectively for $\tilde{Q}_d$.
Let $\{A_i\}_{i=0}^{d}$,$\{A_i^*\}_{i=0}^{d}$,$\{E_i\}_{i=0}^{d}$,$\{E_i^*\}_{i=0}^{d}$ denote the distance matrices, dual distance matrices, primitive idempotents and dual primitive idempotents respectively for Q_d. We can inject the subconstituent algebra $T(x)$ of $\tilde{Q}_d$ into the subconstituent algebra $\tilde{T}(y)$ of Q_d as follows

The antipodal quotient of Q_d, continued

We have, for $0 \leq i \leq h$, $a_i = \frac{1}{2}(A_i + A_{d-i})$, and

$$a_i^* = \begin{cases} A_i^* & i \text{ even} \\ A_{d-i}^* & i \text{ odd.} \end{cases}$$

So $a^* = A_{d-1}^*$.

The antipodal quotient of Q_d, concluded

Because, A, A^*, A^ε act as Y, X, Z, we have that $T(x)$ is an $\mathcal{A}$-module where y, x, z act as $A, (-1)^h sA^*, (-1)^h s\mathbf{i}A^\varepsilon$. We also have that for $0 \leq i \leq h$

$$(-1)^h sA_i^* = \begin{cases} a_i^* & i \text{ even} \\ a_{d-i}^* & i \text{ odd,} \end{cases}$$

so $(-1)^h sA^* = a^*$. This means the Leonard pairs from $\tilde{Q}_d$ also extend to Leonard triples, with imaginary adjacency matrix $a^\varepsilon = (-1)^h s\mathbf{i}A^\varepsilon$.
Furthermore, the application of $(-1)^h s$ to a $T(x)$-module for Q_d splits every irreducible $T(x)$-module into two irreducible $\mathcal{A}$-modules, one of which is an irreducible $\tilde{T}(y)$-module.

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