Symmetric graphs of diameter two with complete normal quotients

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Symmetries of Graphs and Networks II

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General Problem

Investigate the structure of graphs Γ where

- $|V(\Gamma)|$ finite
- $\mathrm{diam}(\Gamma) = 2$
- Γ is symmetric or arc-transitive

Why?

- small diameter - desirable in network design
- includes important families of graphs, e.g. all arc-transitive strongly regular graphs

Normal quotients

Γ, a graph; $N \lhd G \leqslant \mathrm{Aut}\,(\Gamma)$

G-normal quotient of Γ with respect to N : graph Γ_N with

- $V(\Gamma_N)$: N-orbits
- $E(\Gamma_N)$: $\{A, B\}$ such that $\{a, b\} \in E(\Gamma)$ for some $a \in A, b \in B$

Γ_N is nontrivial if N is intransitive on $V(\Gamma)$ and $\Gamma_N \neq \Gamma$.

Normal quotients

Properties of Γ_N

- $\mathrm{diam}(\Gamma) = 2$
  $\Rightarrow \Gamma_N$ complete or $\mathrm{diam}(\Gamma_N) = 2$
- Γ connected, G-arc-transitive
  $\Rightarrow \Gamma_N$ connected, G/N-arc-transitive
- Γ is a k-multicover of Γ_N for some $k \in \mathbb{Z}^+$
  i.e., $A \sim_{\Gamma_N} B \Rightarrow$ each $a \in A$ is adjacent to exactly k elements in B, and vice versa

Normal quotients

Reduction

Γ a G-arc-transitive graph; $\mathrm{diam}(\Gamma) = 2$. Either:

1. $\nexists\ N \lhd G$ with Γ_N nontrivial
   i.e., G acts quasiprimitively on $V(\Gamma)$; or
2. $\exists\ N \lhd G$ with Γ_N nontrivial.
   2.1 All nontrivial Γ_N are complete graphs.
   2.2 $\exists$ nontrivial Γ_N with $\mathrm{diam}(\Gamma_N) = 2$.

If 2.2, set $\Gamma' := \Gamma_N$.
$\Rightarrow \Gamma'$ is G/N-arc-transitive; $\mathrm{diam}(\Gamma') = 2$.
Repeat for Γ' and G/N until we get 1 or 2.1. (basic graphs)

The graphs in this talk satisfy 2.1.

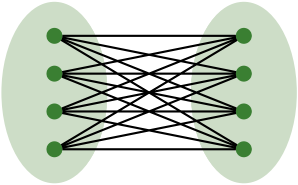
Graphs with complete nontrivial normal quotients

Γ G-arc-transitive, $\mathrm{diam}(\Gamma) = 2$, all nontrivial Γ_N are complete graphs

Example

$\Gamma = K_m \left[\overline{K_n} \right]$ (lexicographic product)

- $V(\Gamma) = V(K_m) \times V(K_n)$
- $(x, y) \sim (x', y') \Leftrightarrow x \neq x'$



$G := S_n \wr S_m$, $N := S_n^m \lhd G$

$\Gamma_N \cong K_m$ (unique nontrivial G-normal quotient)

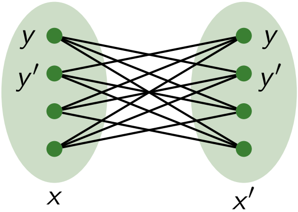
Graphs with complete nontrivial normal quotients

Γ G-arc-transitive, $\mathrm{diam}(\Gamma) = 2$, all nontrivial Γ_N are complete graphs

Example

$\Gamma = K_m \times K_n$ (direct product)

- $V(\Gamma) = V(K_m) \times V(K_n)$
- $(x, y) \sim (x', y') \Leftrightarrow x \neq x'$ and $y \neq y'$



$G := S_m \times S_n$; $\exists$ exactly 2 nontrivial G-normal quotients:
$\Gamma_M \cong K_n$ $(M = S_m)$, $\Gamma_N \cong K_m$ $(N = S_n)$

Graphs with complete nontrivial normal quotients

Γ G-arc-transitive, $\mathrm{diam}(\Gamma) = 2$, all nontrivial Γ_N are complete graphs

CASE: Γ has ≥ 3 distinct nontrivial normal quotients.

Lemma

Let $L, M, N \lhd G$ (minimal normal), such that Γ_L, Γ_M, Γ_N are nontrivial and pairwise distinct. Then:

1. $L \cong M \cong N$ *and* L, M *and* N *are elementary abelian;*
2. $\Gamma_L \cong \Gamma_M \cong \Gamma_N \cong K_{|N|}$;
3. $|V(\Gamma)| = |N|^2$; *and*
4. $L \leqslant M \times N = \mathrm{soc}(G)$, *and* $M \times N$ *acts regularly on* $V(\Gamma)$.

Identify $M, N \leftrightarrow U$, vector space over a finite field

$$M \times N \leftrightarrow V = U \oplus U.$$

Then $\Gamma \cong \mathrm{Cay}(V, S)$ for some $S \subseteq V$, and $G \leqslant \mathrm{AGL}(V)$.

Graphs with complete nontrivial normal quotients

Γ G-arc-transitive, $\mathrm{diam}(\Gamma) = 2$, all nontrivial Γ_N are complete graphs

CASE: Γ has ≥ 3 distinct nontrivial normal quotients.

Theorem

$\Gamma \cong \mathrm{Cay}(V, S)$ and $G \cong T_V \rtimes G_0 \leqslant \mathrm{AGL}(V)$ ($T_V :=$ translations of V), where

- *$V = U \oplus U$, U a vector space over a finite field;*
- *$G_0 = \{(h, h) \mid h \in H\} \leqslant \mathrm{GL}(V)$ for some*
- *$H \leqslant \mathrm{GL}(U)$, transitive on $U \setminus \{\mathbf{0}_U\}$;*
- *$S \subset V$ is G_0-orbit with $\mathbf{0}_V \notin S$, $S = -S$, $\langle S \rangle = V$.*

Conversely, any such graph is connected, G-arc-transitive, and has ≥ 3 nontrivial G-normal quotients, all complete graphs $\cong K_{|U|}$.

(Is $\mathrm{diam}(\Gamma) = 2$?)

Graphs with complete nontrivial normal quotients

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- *$S \subset V$ is G_0-orbit with $\mathbf{0}_V \notin S$, $S = -S$, $\langle S \rangle = V$.*

Minimal normal subgroups of $G = T_V \rtimes G_0$: subgroups of T_V corresponding to $U \oplus \{\mathbf{0}_U\}$, $\{\mathbf{0}_U\} \oplus U$, and $\{(u, u^\varphi) \mid u \in U\}$ for any $\varphi \in C_{\mathrm{GL}(U)}(H) \Rightarrow \exists$ at most $|U| + 1$ distinct nontrivial normal quotients

Graphs with complete nontrivial normal quotients

Example : Γ with ≥ 3 distinct nontrivial normal quotients

$V = U \oplus U,\ G = T_V \rtimes G_0$

- $U :=$ vector space of dimension 6 over $\mathbb{F}_q$, q even
- $H := G_2(q),\ G_0 := \{(h,h) \mid h \in H\}$

Recall :

$G_2(q) \leqslant \mathrm{Sp}(6,q)$; $B :=$ symplectic form

acts on a generalized hexagon $\mathcal{H}(q)$

- has parameters (q,q)
- point set : set of 1-spaces of U
- line set : subset of the set of totally isotropic 2-spaces of U

Graphs with complete nontrivial normal quotients

Example : Γ with ≥ 3 distinct nontrivial normal quotients

$V = U \oplus U,\ G = T_V \rtimes G_0$

- $U :=$ vector space of dimension 6 over $\mathbb{F}_q$
- $H := G_2(q),\ G_0 := \{(h,h) \mid h \in H\}$

G_0-orbits in $V \setminus \{\mathbf{0}_V\}$:

- $U \oplus \{\mathbf{0}_U\}$, $\{\mathbf{0}_U\} \oplus U$, $\{(u, \lambda u) \mid u \in U\}$ for any $\lambda \in \mathbb{F}_q^*$
- $S_\lambda := \{(u,w) \mid B(u,w) = \lambda,\ \dim\langle u,w\rangle = 2\}$ for any $\lambda \in \mathbb{F}_q^*$
- $S_{\mathcal{L}} := \{(u,w) \mid u,w \in U; \langle u,w\rangle \in$ line set of $\mathcal{H}(q)\}$
- $S_{\mathcal{L}'} := \{(u,w) \mid u,w \in U; \langle u,w\rangle$ totally isotropic but not a line of $\mathcal{H}(q)\}$

Graphs with complete nontrivial normal quotients

Example : Γ with ≥ 3 distinct nontrivial normal quotients

$V = U \oplus U$, $G = T_V \rtimes G_0$

- $U :=$ vector space of dimension 6 over $\mathbb{F}_q$
- $H := G_2(q)$, $G_0 := \{(h, h) \mid h \in H\}$

$\Gamma = \mathrm{Cay}(V, S)$ is connected only if $S = S_\lambda$ $(\lambda \in \mathbb{F}_q^*)$, $S_\mathcal{L}$ or $S_{\mathcal{L}'}$.

Theorem

$\mathrm{diam}(\Gamma) = 2$ *if* $S = S_\mathcal{L}$, $S_{\mathcal{L}'}$ *or* S_λ $(\lambda \in \mathbb{F}_q^*)$.

Γ has $(q+1)$ distinct nontrivial G-normal quotients corresponding to $U \oplus \{\mathbf{0}_U\}$, $\{\mathbf{0}_U\} \oplus U$, and $\{(u, \lambda u) \mid u \in U\}$ for each $\lambda \in \mathbb{F}_q^*$; all $\cong K_{q^6}$.

Graphs with complete nontrivial normal quotients

Γ G-arc-transitive, $\mathrm{diam}(\Gamma) = 2$, all nontrivial Γ_N are complete graphs

CASE: Γ has exactly 2 distinct nontrivial normal quotients.

Suppose $M, N \lhd G$ (minimal normal) correspond to the two nontrivial normal quotients.

If $\exists\ L \lhd G$ (minimal normal), then either

- L is transitive on $V(\Gamma)$, or
- $L \cong M$ or N.

Examples:

- direct product $K_m \times K_n$
- (P. Spiga) $\Gamma = \mathrm{Cay}(\mathbb{F}_q \oplus \mathbb{F}_q, S)$, $G = (\mathbb{F}_q \oplus \mathbb{F}_q) \rtimes G_0$ where
  - $G_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \ ab \text{ a square in } \mathbb{F}_q^* \right\}$
  - $S = \{(a, b) \mid ab \text{ a square in } \mathbb{F}_q^*\}$

Graphs with complete nontrivial normal quotients

Γ G-arc-transitive, $\mathrm{diam}(\Gamma) = 2$, all nontrivial Γ_N are complete graphs

CASE: Γ has a unique nontrivial normal quotient Γ_N.

WLOG suppose that N is the stabiliser of its orbits in $V(\Gamma)$ (i.e., N is 1-closed).

Then either

- $N \geqslant \mathrm{soc}(G)$, or
- N acts semiregularly on $V(\Gamma)$.

Examples:

- lexicographic product $K_m \left[\overline{K_n} \right]$
- others?

END

Thank you!